Proof Details for "Existence and Performance of Shalvi-Weinstein Estimators"†

Philip Schniter

† Submitted for publication in *IEEE Transactions on Signal Processing*.

P. Schniter is with Department of Electrical Engineering at The Ohio State University, Columbus OH 43210.

Direct all correspondence to: Phil Schniter, OSU Dept. of Electrical Engineering, 2015 Neil Ave, Columbus, OH, 43210 USA. (614) 247-6488, (614) 292-7596 FAX, *schniter@ee.eng.ohio-state.edu*.

We assume in this document that all quantities are real-valued. Extension to complex-valued quantities is straightforward.

I. Maxima of $|\mathcal{K}_y|$ along the boundary of $\mathcal{C}_\nu^{(0)} \cap \mathcal{S}$.

From the definitions of the dominant cone $\mathcal{C}_\nu^{(0)}$ and the unit sphere $\mathcal{S}$, points $\mathbf{q}$ on the boundary of $\mathcal{C}_\nu^{(0)} \cap \mathcal{S}$ have the properties that $|q_\nu^{(0)}| = \max_{(\ell,\delta)\neq(0,\nu)} |q_\delta^{(\ell)}|$ and $\|\underline{\boldsymbol{q}}\|_2^2 = 1$. For a particular pair of maximum elements $\{q_\nu^{(0)}, q_\delta^{(\ell)}\}$, let us define $\breve{\boldsymbol{q}}$ to be the vector $\underline{\boldsymbol{q}}$ with these two maximum elements omitted. Then since $1 = \|\underline{\boldsymbol{q}}\|_2^2 = 2|q_\nu^{(0)}|^2 + \|\breve{\boldsymbol{q}}\|_2^2$, we know that $|q_\nu^{(0)}| = \sqrt{(1-\|\breve{\boldsymbol{q}}\|_2^2)/2}$. Note that $\breve{\boldsymbol{q}}$ parameterizes the $(\ell,\delta)^{th}$ edge of the $\mathcal{C}_\nu^{(0)} \cap \mathcal{S}$ boundary when the magnitude of the largest coefficient in $\breve{\boldsymbol{q}}$ is at most $|q_\nu^{(0)}|$. (By the "$(\ell,\delta)^{th}$ edge," we mean the boundary between the dominant cones of desired component $q_\nu^{(0)}$ and interference component $q_\delta^{(\ell)}$. The union of the $(\ell,\delta)^{th}$ edges for all $(\ell,\delta) \neq (0,\nu)$ forms the boundary of the desired cone.)

The kurtosis of responses on the $(\ell,\delta)^{th}$ edge can be written

$$\begin{aligned}
\mathcal{K}_y &= \sum_k \|\underline{\boldsymbol{q}}^{(k)}\|_4^4 \mathcal{K}_s^{(k)} \\
&= |q_\nu^{(0)}|^4 \mathcal{K}_s^{(0)} + |q_\delta^{(\ell)}|^4 \mathcal{K}_s^{(\ell)} + \sum_k \|\breve{\boldsymbol{q}}^{(k)}\|_4^4 \mathcal{K}_s^{(k)} \\
&= \left(1 - \|\breve{\boldsymbol{q}}\|_2^2\right)^2 \left(\mathcal{K}_s^{(0)} + \mathcal{K}_s^{(\ell)}\right)/4 + \sum_k \|\breve{\boldsymbol{q}}^{(k)}\|_4^4 \mathcal{K}_s^{(k)} \qquad (1)
\end{aligned}$$

where $\breve{\boldsymbol{q}}^{(k)}$ extracts the elements of $\breve{\boldsymbol{q}}$ corresponding to source k.

Below we evaluate the gradient and Hessian of $\mathcal{K}_y$ to show that there exists a unique local maximum of $|\mathcal{K}_y|$ at the point $\breve{\boldsymbol{q}} = 0$ as long as $\mathcal{K}_s^{(\ell)} \neq -\mathcal{K}_s^{(0)}$. Using straightforward calculus, it is possible to derive the component of the gradient of $\mathcal{K}_y$ in the direction of $q_b^{(a)}$:

$$[\nabla_{\breve{\boldsymbol{q}}} \mathcal{K}_y]_b^{(a)} = \left((\|\breve{\boldsymbol{q}}\|_2^2 - 1)(\mathcal{K}_s^{(0)} + \mathcal{K}_s^{(\ell)}) + 4(q_b^{(a)})^2 \mathcal{K}_s^{(a)}\right) q_b^{(a)} \qquad (2)$$

and component of the Hessian in the directions $q_b^{(a)}$ and $q_d^{(c)}$:

$$[\mathcal{H}_{\breve{\boldsymbol{q}}} \mathcal{K}_y]_{b,d}^{(a,c)} = \begin{cases} (2(q_b^{(a)})^2 + \|\breve{\boldsymbol{q}}\|_2^2 - 1)(\mathcal{K}_s^{(0)} + \mathcal{K}_s^{(\ell)}) + 12(q_b^{(a)})^2 \mathcal{K}_s^{(a)}, & (a,b) = (c,d), \\ 2 q_b^{(a)} q_d^{(c)} (\mathcal{K}_s^{(0)} + \mathcal{K}_s^{(\ell)}), & (a,b) \neq (c,d). \end{cases} \qquad (3)$$

A stationary point occurs when the gradient components obey $[\nabla_{\breve{\boldsymbol{q}}} \mathcal{K}_y]_b^{(a)} = 0 \;\; \forall (a,b) \notin \{(0,\nu),(\ell,\delta)\}$. Hence, from (2),

$$q_b^{(a)}\big|_{\text{stationary}} = \frac{1}{2}\alpha_b^{(a)} \sqrt{(1 - \|\breve{\boldsymbol{q}}\|_2^2)\frac{\mathcal{K}_s^{(0)} + \mathcal{K}_s^{(\ell)}}{\mathcal{K}_s^{(a)}}}, \qquad \text{for} \;\; \alpha_b^{(a)} \in \{-1, 0, 1\}. \qquad (4)$$

Note that for stationary point coefficients $q_b^{(a)} \neq 0$, we require that $0 < \frac{\mathcal{K}_s^{(0)}+\mathcal{K}_s^{(\ell)}}{\mathcal{K}_s^{(a)}} < \infty$. (Recall $\|\breve{\boldsymbol{q}}\|_2^2 \leq 1$.) Solving the family of equations (4) for $\|\breve{\boldsymbol{q}}\|_2^2$, and plugging the result back into (4), we can derive an explicit expression for the coefficients of a stationary point. Using $M^{(k)}$ to denote the number of nonzero gradient coefficients associated with the k^{th} source, it can be shown that

$$q_b^{(a)}\big|_{\text{stationary}} = \alpha_b^{(a)} \left(\sqrt{\frac{4\mathcal{K}_s^{(a)}}{\mathcal{K}_s^{(0)}+\mathcal{K}_s^{(\ell)}} + \mathcal{K}_s^{(a)} \sum_k \frac{M^{(k)}}{\mathcal{K}_s^{(k)}}} \right)^{-1} \tag{5}$$

Evaluating the Hessian at the stationary points, we find from (4) and (3) that

$$[\mathcal{H}_{\breve{\boldsymbol{q}}}\mathcal{K}_y]_{b,d}^{(a,c)}\big|_{\text{stationary}} = \begin{cases} \frac{1}{2}(1-\|\breve{\boldsymbol{q}}\|_2^2)(\mathcal{K}_s^{(0)}+\mathcal{K}_s^{(\ell)}) \left((\alpha_b^{(a)})^2 \left(\frac{\mathcal{K}_s^{(0)}+\mathcal{K}_s^{(\ell)}}{\mathcal{K}_s^{(a)}} + 6 \right) - 2 \right) & (a,b)=(c,d), \\ \frac{1}{2}(1-\|\breve{\boldsymbol{q}}\|_2^2)(\mathcal{K}_s^{(0)}+\mathcal{K}_s^{(\ell)}) \left(\alpha_b^{(a)}\alpha_d^{(c)} \frac{\mathcal{K}_s^{(0)}+\mathcal{K}_s^{(\ell)}}{\sqrt{\mathcal{K}_s^{(a)}\mathcal{K}_s^{(c)}}} \right) & (a,b)\neq(c,d). \end{cases} \tag{6}$$

Note that $\sqrt{\mathcal{K}_s^{(a)}\mathcal{K}_s^{(c)}}$ is positive when $\alpha_b^{(a)}$ and $\alpha_d^{(c)}$ are both nonzero: since we previously required that $\infty > \frac{\mathcal{K}_s^{(0)}+\mathcal{K}_s^{(\ell)}}{\mathcal{K}_s^{(a)}} > 0$ and $\infty > \frac{\mathcal{K}_s^{(0)}+\mathcal{K}_s^{(\ell)}}{\mathcal{K}_s^{(c)}} > 0$, it follows that $\mathcal{K}_s^{(a)}$ and $\mathcal{K}_s^{(c)}$ must be nonzero with the same sign.

For a stationary point to be a local maximum (minimum), it must have a negative (positive) definite Hessian. Furthermore, a matrix is ND (PD) if and only if all of its principle minors are ND (PD). Thus, we are motivated to consider the 2×2 Hessian minors. From (6), they have the form

$$\frac{1}{2}(1-\|\breve{\boldsymbol{q}}\|_2^2)(\mathcal{K}_s^{(0)}+\mathcal{K}_s^{(\ell)}) \begin{bmatrix} (\alpha_b^{(a)})^2 \left(\frac{\mathcal{K}_s^{(0)}+\mathcal{K}_s^{(\ell)}}{\mathcal{K}_s^{(a)}} + 6 \right) - 2 & \alpha_b^{(a)}\alpha_d^{(c)} \frac{\mathcal{K}_s^{(0)}+\mathcal{K}_s^{(\ell)}}{\sqrt{\mathcal{K}_s^{(a)}\mathcal{K}_s^{(c)}}} \\ \alpha_b^{(a)}\alpha_d^{(c)} \frac{\mathcal{K}_s^{(0)}+\mathcal{K}_s^{(\ell)}}{\sqrt{\mathcal{K}_s^{(a)}\mathcal{K}_s^{(c)}}} & (\alpha_d^{(c)})^2 \left(\frac{\mathcal{K}_s^{(0)}+\mathcal{K}_s^{(\ell)}}{\mathcal{K}_s^{(c)}} + 6 \right) - 2. \end{bmatrix} \tag{7}$$

Consider the three cases:

1. $\underline{q_b^{(a)} = q_d^{(c)} = 0}$: Setting $\{\alpha_b^{(a)}, \alpha_d^{(c)}\} = \{0, 0\}$, the matrix in (7) takes the form

   $$\frac{1}{2}(1-\|\breve{\boldsymbol{q}}\|_2^2)(\mathcal{K}_s^{(0)}+\mathcal{K}_s^{(\ell)}) \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix},$$

   which is ND when $\mathcal{K}_s^{(0)}+\mathcal{K}_s^{(\ell)} > 0$ and PD when $\mathcal{K}_s^{(0)}+\mathcal{K}_s^{(\ell)} < 0$.

2. $\underline{q_b^{(a)} \neq 0,\ q_d^{(c)} = 0}$: Setting $\{\alpha_b^{(a)}, \alpha_d^{(c)}\} = \{\pm 1, 0\}$, the determinant of (7) equals

   $$-\frac{1}{2}(1-\|\breve{\boldsymbol{q}}\|_2^2)^2(\mathcal{K}_s^{(0)}+\mathcal{K}_s^{(\ell)})^2 \left(4 + \underbrace{\frac{\mathcal{K}_s^{(0)}+\mathcal{K}_s^{(\ell)}}{\mathcal{K}_s^{(a)}}}_{\geq 0} \right),$$

   which is negative (assuming $\mathcal{K}_s^{(0)}+\mathcal{K}_s^{(\ell)} \neq 0$) implying (7) is indefinite.

3. $\underline{q_b^{(a)} \neq 0,\ q_d^{(c)} \neq 0}$: Setting $\{\alpha_b^{(a)}, \alpha_d^{(c)}\} = \{\pm 1, 1\}$, the determinant of (7) equals

$$\frac{1}{4}(1 - \|\breve{\boldsymbol{q}}\|_2^2)^2 (\mathcal{K}_s^{(0)} + \mathcal{K}_s^{(\ell)})^2 \left(4 + \frac{\mathcal{K}_s^{(0)} + \mathcal{K}_s^{(\ell)}}{\mathcal{K}_s^{(a)}} + \frac{\mathcal{K}_s^{(0)} + \mathcal{K}_s^{(\ell)}}{\mathcal{K}_s^{(c)}}\right),$$

which is positive (assuming $\mathcal{K}_s^{(0)} + \mathcal{K}_s^{(\ell)} \neq 0$), hence (7) is either ND or PD. Noting that the elements on the diagonal are positive when $\mathcal{K}_s^{(0)} + \mathcal{K}_s^{(\ell)} > 0$ and negative when $\mathcal{K}_s^{(0)} + \mathcal{K}_s^{(\ell)} < 0$, we see that (7) will be PD when $\mathcal{K}_s^{(0)} + \mathcal{K}_s^{(\ell)} > 0$ and ND when $\mathcal{K}_s^{(0)} + \mathcal{K}_s^{(\ell)} < 0$.

Note from (4) that $\mathcal{K}_s^{(0)} + \mathcal{K}_s^{(\ell)} = 0$ implies $\breve{\boldsymbol{q}} = 0$, thus the assumption that $\mathcal{K}_s^{(0)} + \mathcal{K}_s^{(\ell)} \neq 0$ in cases 2 and 3 above is justified. From the three cases above, we see that the only $\breve{\boldsymbol{q}}$ locally maximizing/minimizing $\mathcal{K}_y$ are $\breve{\boldsymbol{q}} = 0$ and $\breve{\boldsymbol{q}}$ with strictly nonzero elements. Taking the case $\mathcal{K}_s^{(0)} + \mathcal{K}_s^{(\ell)} > 0$, the point $\breve{\boldsymbol{q}} = 0$ yields a local $\mathcal{K}_y$ maximum of $(\mathcal{K}_s^{(0)} + \mathcal{K}_s^{(\ell)})/4$ according to (1). When $\mathcal{K}_s^{(0)} + \mathcal{K}_s^{(\ell)} < 0$, the point $\breve{\boldsymbol{q}} = 0$ yields a local $\mathcal{K}_y$ minimum of $(\mathcal{K}_s^{(0)} + \mathcal{K}_s^{(\ell)})/4$. In either case, $|\mathcal{K}_y|$ attains a local maximum at $\breve{\boldsymbol{q}} = 0$. (Using similar arguments, it can be shown that $\breve{\boldsymbol{q}}$ with strictly nonzero elements yields a $|\mathcal{K}_y|$ local minimum.) To conclude, the local $|\mathcal{K}_y|$ maximum over the $(\ell, \delta)^{th}$ edge of the $\mathcal{C}_\nu^{(0)} \cap \mathcal{S}$ boundary occurs at $\breve{\boldsymbol{q}} = 0$ as long as $\mathcal{K}_s^{(0)} + \mathcal{K}_s^{(\ell)} \neq 0$. (In the case $\mathcal{K}_s^{(0)} + \mathcal{K}_s^{(\ell)} = 0$, it is easy to see from (1) that $\breve{\boldsymbol{q}} = 0$ gives $\mathcal{K}_y = 0$, and so $\breve{\boldsymbol{q}} = 0$ must be a local minimum of the non-negative quantity $|\mathcal{K}_y|$.)

So far we have determined the local $|\mathcal{K}_y|$ maximum in the $\breve{\boldsymbol{q}}$ space. But earlier we specified that the valid region of $\breve{\boldsymbol{q}}$ is constrained to vectors whose largest element has magnitude of at most $|q_\nu^{(0)}|$. Hence, there is a possibility that the maximum value of $|\mathcal{K}_y|$ might not be attained at the *local* maximum of our desired region, but rather on the boundary of our desired region in the $\breve{\boldsymbol{q}}$ space. (The maximum of $|\mathcal{K}_y|$ will definitely be attained on the boundary of the $\breve{\boldsymbol{q}}$ region when $\mathcal{K}_s^{(0)} + \mathcal{K}_s^{(\ell)} = 0$, where we found that no local maxima exist.) These boundary points have the property that there exists some pair $(m, n) \notin \{(0, \nu), (\ell, \delta)\}$ such that $|q_n^{(m)}| = |q_\nu^{(0)}| = |q_\delta^{(\ell)}|$. We can parameterize the boundary of the $\breve{\boldsymbol{q}}$ region using $\breve{\breve{\boldsymbol{q}}}$, where $\breve{\breve{\boldsymbol{q}}}$ if formed by removing the coefficients $q_n^{(m)}, q_\nu^{(0)}$ and $q_\delta^{(\ell)}$ from $\underline{\boldsymbol{q}}$. But writing

$$\begin{aligned} \mathcal{K}_y &= |q_\nu^{(0)}|^4 \mathcal{K}_s^{(0)} + |q_\delta^{(\ell)}|^4 \mathcal{K}_s^{(\ell)} + |q_n^{(m)}|^4 \mathcal{K}_s^{(m)} + \sum_k \|\breve{\breve{\boldsymbol{q}}}^{(k)}\|_4^4 \mathcal{K}_s^{(k)} \\ &= \left(1 - \|\breve{\boldsymbol{q}}\|_2^2\right)^2 \left(\mathcal{K}_s^{(0)} + \mathcal{K}_s^{(\ell)} + \mathcal{K}_s^{(m)}\right)/9 + \sum_k \|\breve{\breve{\boldsymbol{q}}}^{(k)}\|_4^4 \mathcal{K}_s^{(k)} \end{aligned} \tag{8}$$

and noticing the similarities between (8) and (1), it is evident that the search for $|\mathcal{K}_y|$ maxima over $\breve{\breve{\boldsymbol{q}}}$ is analogous to the search over $\breve{\boldsymbol{q}}$: we get a local maximum of $|\mathcal{K}_y| = |\mathcal{K}_s^{(0)} + \mathcal{K}_s^{(\ell)} + \mathcal{K}_s^{(m)}|/9$ at $\breve{\breve{\boldsymbol{q}}} = 0$, with the possibility for a non-local maximum on the boundary of valid $\breve{\breve{\boldsymbol{q}}}$. This process (of searching over the boundary of the boundary of the boundary) repeats itself until we have no more free parameters. At the end, we have a family of candidate points $\{\underline{\boldsymbol{q}}\}$ with $M = 2, 3, 4, \ldots$ nonzero coefficients of equal magnitude $\sqrt{1/M}$, where the set of nonzero indices includes $(0, \nu)$. Using $M^{(k)}$ to denote the number of nonzero coefficients associated with the k^{th} source (so that $M = \sum_k M^{(k)}$), these candidate maxima have absolute kurtosis

$$\begin{aligned}
|\mathcal{K}_y| &= \left| \sum_k \sum_i |q_i^{(k)}|^4 \mathcal{K}_s^{(k)} \right| \\
&= \left| \sum_k M^{(k)} (\sqrt{1/M})^4 \mathcal{K}_s^{(k)} \right| \\
&= \left| \frac{1}{M^2} \sum_k M^{(k)} \mathcal{K}_s^{(k)} \right| \\
&= \frac{1}{M^2} \left| \mathcal{K}_s^{(0)} + \underbrace{(M^{(0)} - 1)}_{\geq 0} \mathcal{K}_s^{(0)} + \sum_{k \neq 0} M^{(k)} \mathcal{K}_s^{(k)} \right| \\
&\leq \frac{1}{M^2} \left| \mathcal{K}_s^{(0)} \right| + \frac{M-1}{M^2} \max_k \left| \mathcal{K}_s^{(k)} \right| \\
&\leq \frac{1}{4} \left(\left| \mathcal{K}_s^{(0)} \right| + \max_k \left| \mathcal{K}_s^{(k)} \right| \right)
\end{aligned}$$

where the last step follows from the restriction $M \geq 2$. To conclude,

$$\max_{\underline{\boldsymbol{q}} \in \text{bndr}(\mathcal{C}_\nu^{(0)} \cap \mathcal{S})} |\mathcal{K}_y(\underline{\boldsymbol{q}})| \leq \frac{1}{4} \left(\left| \mathcal{K}_s^{(0)} \right| + \max_k \left| \mathcal{K}_s^{(k)} \right| \right) \tag{9}$$

II. Suprema of $|\mathcal{K}_y|$ in $\mathcal{C}_\nu^{(0)} \cap \mathcal{S}$.

Parameterizing $\underline{\boldsymbol{q}} \in \mathcal{C}_\nu^{(0)} \cap \mathcal{S}$ by $\bar{\boldsymbol{q}}$, which is $\underline{\boldsymbol{q}}$ with the element $q_\nu^{(0)}$ removed, we have

$$\begin{aligned}
\mathcal{K}_y &= |q_\nu^{(0)}|^4 \mathcal{K}_s^{(0)} + \sum_k \|\bar{\boldsymbol{q}}^{(k)}\|_4^4 \, \mathcal{K}_s^{(k)} \\
&= \left(1 - \|\bar{\boldsymbol{q}}\|_2^2\right)^2 \mathcal{K}_s^{(0)} + \sum_k \|\bar{\boldsymbol{q}}^{(k)}\|_4^4 \, \mathcal{K}_s^{(k)}
\end{aligned} \tag{10}$$

which is exactly (1) if we make the substitutions $\mathcal{K}_s^{(\ell)} \to 3\mathcal{K}_s^{(0)}$ and $\breve{\boldsymbol{q}} \to \bar{\boldsymbol{q}}$. Thus, the results of the previous section imply that there exists a unique local $|\mathcal{K}_y|$ maximum at $\bar{\boldsymbol{q}} = 0$ (always, since $\mathcal{K}_s^{(0)} \neq 0$) attaining the value $|\mathcal{K}_s^{(0)}|$. Note that $\bar{\boldsymbol{q}} = 0$ corresponds to $\underline{\boldsymbol{q}} = \mathbf{e}_\nu^{(0)}$.

We must also consider whether larger values of $|\mathcal{K}_y|$ are attained on the boundary of the open set $\mathcal{C}_\nu^{(0)} \cap \mathcal{S}$. Recalling that (9) gives an upper bound for $|\mathcal{K}_y|$ on the boundary, if

$$|\mathcal{K}_s^{(0)}| > \frac{1}{4} \left(\left| \mathcal{K}_s^{(0)} \right| + \max_k \left| \mathcal{K}_s^{(k)} \right| \right),$$

we can be sure that the supremum of $|\mathcal{K}_y|$ over $\mathcal{C}_\nu^{(0)} \cap \mathcal{S}$ is attained at the point $\underline{\boldsymbol{q}} = \mathbf{e}_\nu^{(0)}$ with value $|\mathcal{K}_s^{(0)}|$.

III. Suprema of $|\mathcal{K}_y|$ over $\{\bar{\boldsymbol{q}} \in \bar{\mathcal{Q}}_{\mathcal{C}\cap\mathcal{S}} : \|\bar{\boldsymbol{q}}\|_2^2 > \frac{1}{2}\}$

To determine the suprema of $|\mathcal{K}_y|$ over

$$\bar{\mathcal{Q}}_{\mathcal{C}\cap\mathcal{S},>0.5} \;:=\; \{\bar{\boldsymbol{q}} \in \bar{\mathcal{Q}}_{\mathcal{C}\cap\mathcal{S}} : \|\bar{\boldsymbol{q}}\|_2^2 > \frac{1}{2}\},$$

we first consider the interior of $\bar{\mathcal{Q}}_{\mathcal{C}\cap\mathcal{S},>0.5}$. Since we have shown (in the previous section) that the unique local maximum within $\bar{\mathcal{Q}}_{\mathcal{C}\cap\mathcal{S}}$ occurs at $\bar{\boldsymbol{q}} = 0$, there exists no local maximum within $\bar{\mathcal{Q}}_{\mathcal{C}\cap\mathcal{S},>0.5}$, and thus the supremum over $\bar{\mathcal{Q}}_{\mathcal{C}\cap\mathcal{S},>0.5}$ is attained on the boundary. Since

$$\mathrm{bndr}(\bar{\mathcal{Q}}_{\mathcal{C}\cap\mathcal{S},>0.5}) \;=\; \mathrm{bndr}(\bar{\mathcal{Q}}_{\mathcal{C}\cap\mathcal{S}}) \cup \{\bar{\boldsymbol{q}} : \|\bar{\boldsymbol{q}}\|_2^2 = \frac{1}{2}\}$$

the $|\mathcal{K}_y|$ supremum over $\mathrm{bndr}(\bar{\mathcal{Q}}_{\mathcal{C}\cap\mathcal{S},>0.5})$ will either occur on the boundary of $\bar{\mathcal{Q}}_{\mathcal{C}\cap\mathcal{S}}$ or in $\{\bar{\boldsymbol{q}} : \|\bar{\boldsymbol{q}}\|_2^2 = \frac{1}{2}\}$.

Since we have already examined $|\mathcal{K}_y|$ on $\mathrm{bndr}(\bar{\mathcal{Q}}_{\mathcal{C}\cap\mathcal{S}})$, we now focus on $\mathcal{K}_y(\bar{\boldsymbol{q}})$ when $\|\bar{\boldsymbol{q}}\|_2^2 = \frac{1}{2}$. Such $\bar{\boldsymbol{q}}$ can be parameterized by $\breve{\boldsymbol{q}}$, defined as $\underline{\boldsymbol{q}}$ with elements $q_\nu^{(0)}$ and $q_\delta^{(\ell)}$ omitted, for all combinations $(\ell, \delta) \neq (0, \nu)$. Since $1 = \|\underline{\boldsymbol{q}}\|_2^2 = |q_\nu^{(0)}|^2 + \|\bar{\boldsymbol{q}}\|_2^2 = |q_\nu^{(0)}|^2 + \frac{1}{2}$, we have $|q_\nu^{(0)}|^2 = \frac{1}{2}$, and since $\frac{1}{2} = \|\bar{\boldsymbol{q}}\|_2^2 = |q_\delta^{(\ell)}|^2 + \|\breve{\boldsymbol{q}}\|_2^2$, we have $|q_\delta^{(\ell)}|^2 = \frac{1}{2} - \|\breve{\boldsymbol{q}}\|_2^2$. Thus

$$\begin{aligned}
\mathcal{K}_y &= \sum_k \|\underline{\boldsymbol{q}}^{(k)}\|_4^4 \mathcal{K}_s^{(k)} \\
&= |q_\nu^{(0)}|^4 \mathcal{K}_s^{(0)} + |q_\delta^{(\ell)}|^4 \mathcal{K}_s^{(\ell)} + \sum_k \|\breve{\boldsymbol{q}}^{(k)}\|_4^4 \mathcal{K}_s^{(k)} \\
&= \frac{1}{4}\mathcal{K}_s^{(0)} + \left(\frac{1}{2} - \|\breve{\boldsymbol{q}}\|_2^2\right)^2 \mathcal{K}_s^{(\ell)} + \sum_k \|\breve{\boldsymbol{q}}^{(k)}\|_4^4 \mathcal{K}_s^{(k)}.
\end{aligned} \tag{11}$$

We now evaluate the gradient and Hessian of $\mathcal{K}_y(\breve{\boldsymbol{q}})$. From (11), the $\mathcal{K}_y$ gradient component in the direction of $q_b^{(a)}$ equals

$$[\nabla_{\breve{\boldsymbol{q}}} \mathcal{K}_y]_b^{(a)} \;=\; 4\left((\|\breve{\boldsymbol{q}}\|_2^2 - \frac{1}{2})\mathcal{K}_s^{(\ell)} + (q_b^{(a)})^2 \mathcal{K}_s^{(a)} \right) q_b^{(a)} \tag{12}$$

and Hessian component in the directions $q_b^{(a)}$ and $q_d^{(c)}$ equals

$$\frac{1}{4}\,[\mathcal{H}_{\breve{\boldsymbol{q}}} \mathcal{K}_y]_{b,d}^{(a,c)} \;=\; \begin{cases} (2(q_b^{(a)})^2 + \|\breve{\boldsymbol{q}}\|_2^2 - \frac{1}{2})\mathcal{K}_s^{(\ell)} + 3(q_b^{(a)})^2 \mathcal{K}_s^{(a)}, & (a,b) = (c,d), \\ 2 q_b^{(a)} q_d^{(c)} \mathcal{K}_s^{(\ell)}, & (a,b) \neq (c,d). \end{cases} \tag{13}$$

A stationary point occurs when the gradient components obey $[\nabla_{\breve{\boldsymbol{q}}}\mathcal{K}_y]_b^{(a)} = 0 \;\; \forall (a,b) \notin \{(0,\nu),(\ell,\delta)\}$. Hence, from (2),

$$q_b^{(a)}\big|_{\text{stationary}} = \alpha_b^{(a)} \sqrt{(\frac{1}{2} - \|\breve{\boldsymbol{q}}\|_2^2)\frac{\mathcal{K}_s^{(\ell)}}{\mathcal{K}_s^{(a)}}}, \qquad \text{for} \;\; \alpha_b^{(a)} \in \{-1,0,1\}. \tag{14}$$

Note that for stationary point coefficients $q_b^{(a)} \neq 0$, we require that $0 < \frac{\mathcal{K}_s^{(\ell)}}{\mathcal{K}_s^{(a)}} < \infty$. (Recall $\|\breve{\boldsymbol{q}}\|_2^2 \leq \frac{1}{2}$.) Evaluating the Hessian at the stationary points, we find from (14) and (13) that

$$\frac{1}{4}\,[\mathcal{H}_{\breve{\boldsymbol{q}}}\mathcal{K}_y]_{b,d}^{(a,c)}\big|_{\text{stationary}} = \begin{cases} (\frac{1}{2} - \|\breve{\boldsymbol{q}}\|_2^2)\mathcal{K}_s^{(\ell)}\left((\alpha_b^{(a)})^2\left(2\frac{\mathcal{K}_s^{(\ell)}}{\mathcal{K}_s^{(a)}}+3\right)-1\right) & (a,b)=(c,d), \\ (\frac{1}{2} - \|\breve{\boldsymbol{q}}\|_2^2)\mathcal{K}_s^{(\ell)}\left(2\alpha_b^{(a)}\alpha_d^{(c)}\frac{\mathcal{K}_s^{(\ell)}}{\sqrt{\mathcal{K}_s^{(a)}\mathcal{K}_s^{(c)}}}\right) & (a,b)\neq(c,d). \end{cases} \tag{15}$$

Note that $\sqrt{\mathcal{K}_s^{(a)}\mathcal{K}_s^{(c)}}$ is positive when $\alpha_b^{(a)}$ and $\alpha_d^{(c)}$ are both nonzero: since $\frac{\mathcal{K}_s^{(\ell)}}{\mathcal{K}_s^{(a)}} > 0$ and $\frac{\mathcal{K}_s^{(\ell)}}{\mathcal{K}_s^{(c)}} > 0$, then $\mathcal{K}_s^{(a)}$ and $\mathcal{K}_s^{(c)}$ must be nonzero with the same sign. As before, we examine the 2×2 principle minors:

1. $\underline{q_b^{(a)} = q_d^{(c)} = 0}$: Using $\{\alpha_b^{(a)}, \alpha_d^{(c)}\} = \{0,0\}$, the matrix takes the form

$$4(\frac{1}{2} - \|\breve{\boldsymbol{q}}\|_2^2)\mathcal{K}_s^{(\ell)}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

   which is ND when $\mathcal{K}_s^{(\ell)} > 0$ and PD when $\mathcal{K}_s^{(\ell)} < 0$.

2. $\underline{q_b^{(a)} \neq 0, q_d^{(c)} = 0}$: Using $\{\alpha_b^{(a)}, \alpha_d^{(c)}\} = \{\pm 1, 0\}$, the matrix takes the form

$$8(\frac{1}{2} - \|\breve{\boldsymbol{q}}\|_2^2)\mathcal{K}_s^{(\ell)}\begin{bmatrix} \frac{\mathcal{K}_s^{(\ell)}}{\mathcal{K}_s^{(a)}}+1 & 0 \\ 0 & -1 \end{bmatrix},$$

   which is indefinite (assuming $\mathcal{K}_s^{(\ell)} \neq 0$).

3. $\underline{q_b^{(a)} \neq 0, q_d^{(c)} \neq 0}$: Using $\{\alpha_b^{(a)}, \alpha_d^{(c)}\} = \{\pm 1, 1\}$, the determinant equals

$$16(\frac{1}{2} - \|\breve{\boldsymbol{q}}\|_2^2)^2(\mathcal{K}_s^{(\ell)})^2\left(1 + \frac{\mathcal{K}_s^{(\ell)}}{\mathcal{K}_s^{(a)}} + \frac{\mathcal{K}_s^{(\ell)}}{\mathcal{K}_s^{(c)}}\right),$$

   which is positive (assuming $\mathcal{K}_s^{(\ell)} \neq 0$), hence the matrix is either ND or PD. Noting from (15) that the elements on the diagonal will be positive when $\mathcal{K}_s^{(\ell)} > 0$ and negative when $\mathcal{K}_s^{(\ell)} < 0$, we see that the minor will be PD when $\mathcal{K}_s^{(\ell)} > 0$ and ND when $\mathcal{K}_s^{(\ell)} < 0$.

Since $\mathcal{K}_s^{(\ell)} = 0$ implies $\breve{\boldsymbol{q}} = 0$, the assumption that $\mathcal{K}_s^{(\ell)} \neq 0$ in cases 2 and 3 above is justified. From the three cases above, we see that the only $\breve{\boldsymbol{q}}$ locally maximizing/minimizing $\mathcal{K}_y$ are $\breve{\boldsymbol{q}} = 0$ and $\breve{\boldsymbol{q}}$ with strictly nonzero elements. Taking the case $\mathcal{K}_s^{(\ell)} > 0$, the point $\breve{\boldsymbol{q}} = 0$ yields a local $\mathcal{K}_y$

maximum of $(\mathcal{K}_s^{(0)} + \mathcal{K}_s^{(\ell)})/4$ according to (11). When $\mathcal{K}_s^{(\ell)} < 0$, the point $\breve{\boldsymbol{q}} = 0$ yields a local $\mathcal{K}_y$ minimum of $(\mathcal{K}_s^{(0)} + \mathcal{K}_s^{(\ell)})/4$. In either case, $|\mathcal{K}_y|$ attains a local maximum of $|\mathcal{K}_s^{(0)} + \mathcal{K}_s^{(\ell)}|/4$ at $\breve{\boldsymbol{q}} = 0$. (Using similar arguments, it can be shown that $\breve{\boldsymbol{q}}$ with strictly nonzero elements yields a $|\mathcal{K}_y|$ local minimum.) If we compare these maxima over all choices (ℓ, δ), we find that $|\mathcal{K}_y|$ is upper bounded by $|\mathcal{K}_s^{(0)}|/4 + \max_k |\mathcal{K}_s^{(k)}|/4$.

To conclude, the supremum of $|\mathcal{K}_y|$ over $\bar{\mathcal{Q}}_{\mathcal{C}\cap\mathcal{S},>0.5}$ occurs on the boundary of $\bar{\mathcal{Q}}_{\mathcal{C}\cap\mathcal{S},>0.5}$ and attains a value of at most $|\mathcal{K}_s^{(0)}|/4 + \max_k |\mathcal{K}_s^{(k)}|/4$. Note that, since $\bar{\mathcal{Q}}_{\mathcal{C}\cap\mathcal{S},>0.5}$ is open, the supremum exists outside the set.