Approximate Message Passing: Applications to Communications Receivers

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The Generalized Linear Model:

- Consider observation $\boldsymbol{y} \in \mathbb{C}^M$ of unknown vector $\boldsymbol{x} \in \mathbb{C}^N$ that is
  - sent through known linear transform $\boldsymbol{A}$, generating hidden $\boldsymbol{z} = \boldsymbol{A}\boldsymbol{x}$, then
  - observed through a probabilistic measurement channel $p_{\mathsf{y}|\mathsf{z}}(\boldsymbol{y}|\boldsymbol{z})$.

  Our goal is to infer $\boldsymbol{x}$ from $\boldsymbol{y}$.

- When p_{x} and $p_{\mathsf{y}|\mathsf{z}}$ are both Gaussian, the MMSE/MAP estimator is linear and easy to state in closed-form. The more interesting case is when p_{x} and/or $p_{\mathsf{y}|\mathsf{z}}$ are non-Gaussian.

- Equally interesting is when $M \ll N$: Compressive sensing tells us that K-*sparse* $\boldsymbol{x} \in \mathbb{C}^N$ can be accurately recovered from $M \gtrsim O(K \log N/K)$ measurements when $\boldsymbol{A}$ is information-preserving (e.g., satisfies $2K$-RIP).

- There are many applications of estimation under the generalized linear model in engineering, biology, medicine, finance, etc.

Example Applications:

- Pilot-aided channel estimation / "compressed channel sensing"

  $\boldsymbol{x}$: sparse channel impulse response (length N)

  $\boldsymbol{y}$: pilot observations ($M < N$ with sparse channel)

  $\boldsymbol{A}$: built from pilot symbols and other aspects of linear-modulation

- Imaging (medical, radar, etc.)

  $\boldsymbol{x}$: spatial-domain image (rasterized)

  $\boldsymbol{y}$: noisy measurements (AWGN, Gaussian, phaseless, etc.)

  $\boldsymbol{A}$: typically Fourier-based (details are application dependent)

- Binary linear classification and feature selection

  $\boldsymbol{x}$: prediction vector ($\perp$ to class-separating hyperplane, sparse)

  $\boldsymbol{y}$: binary experimental outcomes (e.g., $\{\text{sick}, \text{healthy}\}$)

  $\boldsymbol{A}$: each row contains per-experient features (e.g., age, weight, etc.)

Generalized Approximate Message Passing (GAMP):

- Suppose we are interested in computing the MMSE or MAP estimate of $\boldsymbol{x}$ from $\boldsymbol{y}$ (under known $\boldsymbol{A}$, p_{x}, $p_{\mathsf{y}|\mathsf{z}}$).
- For general $\boldsymbol{A}$, p_{x}, and $p_{\mathsf{y}|\mathsf{z}}$, this is difficult. . . in fact NP hard.
- However, for sufficiently large and dense $\boldsymbol{A}$, and separable p_{x} and $p_{\mathsf{y}|\mathsf{z}}$, there is a remarkable new iterative algorithm that gets close: GAMP.

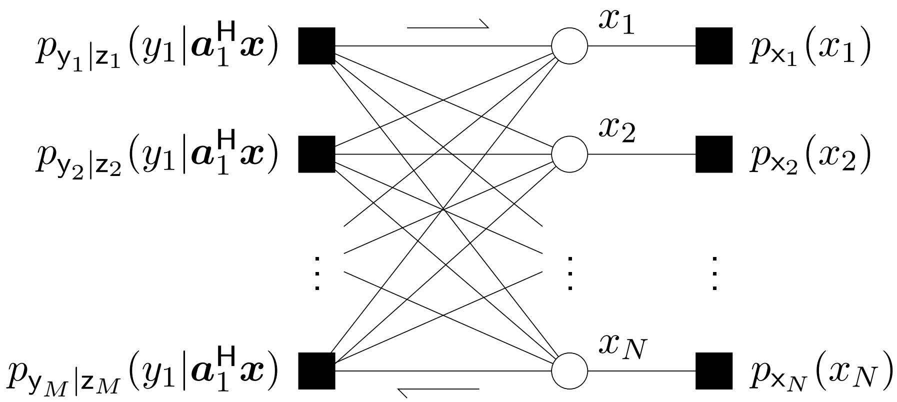
  S. Rangan, "Generalized approximate message passing for estimation with random linear mixing," *arXiv:1010.5141*, Oct. 2010.

  – In the large-system limit ($M, N \to \infty$ with fixed M/N) when $\boldsymbol{A}$ is drawn iid sub-Gaussian, and p_{x} and $p_{\mathsf{y}|\mathsf{z}}$ are separable (i.e., independent r.v.s), GAMP's performance is characterized by a *state evolution* whose fixed points, when unique, coincide with the MMSE or MAP optimal estimates.

  – In practice, $\boldsymbol{A}$ is finite sized and structured (e.g., Fourier). Still, for any $\boldsymbol{A}$, the fixed-points of the GAMP iterations correspond to the critical points of the MAP optimization objective, $\max_{\boldsymbol{x}} \{ \ln p_{\mathsf{y}|\mathsf{z}}(\boldsymbol{y}|\boldsymbol{A}\boldsymbol{x}) + \ln p_{\mathsf{x}}(\boldsymbol{x}) \}$.

A Revolution in Loopy Belief Propagation:

- The GAMP algorithm can be derived as an approximation of the sum-product (in the MMSE case) or max-product (in the MAP case) loopy BP algorithms.



- The approximation makes use of the central limit theorem and Taylor series approximations that hold in the large-system limit.

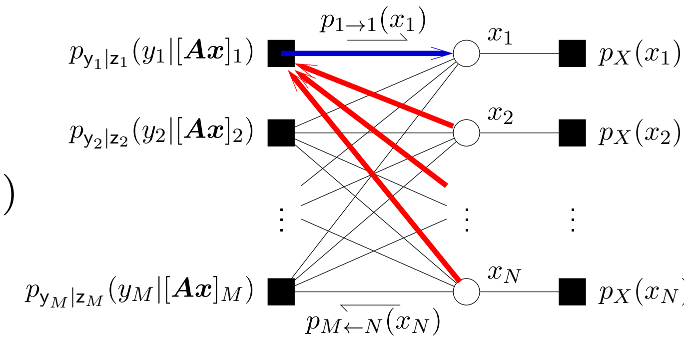
- An interesting observation is that, because $\boldsymbol{A}$ is dense, the factor graph is extremely loopy. Loosely speaking, these loops are OK because (for normalized $\boldsymbol{A}$) they get "weaker" as the problem gets larger.

- Note: Rigorous analyses of GAMP are based on the algorithm itself, not on the loopy-BP approximations.

  M. Bayati and A. Montanari, "The dynamics of message passing on dense graphs, with applications to compressed sensing," *IEEE Trans. Inform. Thy.*, Feb. 2011.

GAMP Heuristics (Sum-Product Case):

1. Message from y_i node to x_j node:



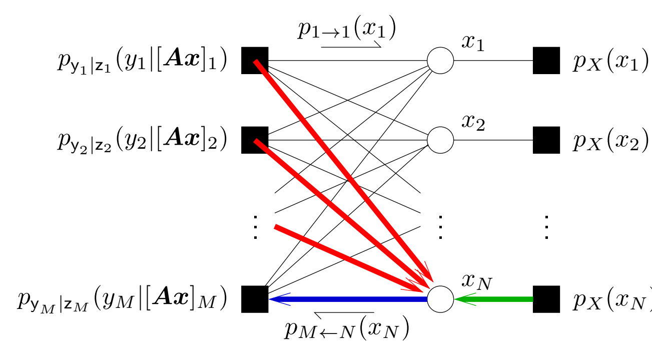
$$p_{i\to j}(x_j) \propto \int_{\{x_r\}_{r\neq j}} p_{\mathsf{y}_i|\mathsf{z}_i}\Big(y_i \Big| \overbrace{\textstyle\sum_r a_{ir}x_r}^{\approx\,\mathcal{N}\text{ via CLT}}\Big) \prod_{r\neq j} p_{i\leftarrow r}(x_r)$$

$$\approx \int_{z_i} p_{\mathsf{y}_i|\mathsf{z}_i}(y_i|z_i)\,\mathcal{N}\big(z_i;\hat{z}_i(x_j),\nu_i^z(x_j)\big) \approx \mathcal{N}$$

To compute $\hat{z}_i(x_j), \nu_i^z(x_j)$, the means and variances of $\{p_{i\leftarrow r}\}_{r\neq j}$ suffice, thus Gaussian message passing!

Remaining problem: we have $2MN$ messages to compute (too many!).

2. Exploiting similarity among the messages $\{p_{i\leftarrow j}\}_{i=1}^M$, AMP employs a Taylor-series approximation of their difference whose error vanishes as $M\to\infty$ for dense $\boldsymbol{A}$ (and similar for $\{p_{i\to j}\}_{j=1}^N$ as $N\to\infty$). Finally, need to compute only $\mathcal{O}(M+N)$ messages!



The resulting algorithm requires two matrix-vector multiplications per iteration, and converges in typically $\lesssim 25$ iterations.

GAMP Extensions:

- Standard GAMP assumes known, separable $p_{\mathbf{x}}$ and $p_{\mathbf{y}|\mathbf{z}}$.
- However, in practice...
  - Densities $p_{\mathbf{x}}$ and $p_{\mathbf{y}|\mathbf{z}}$ are usually unknown.
  - Often, they are also non-separable (i.e., elements of $\mathbf{x}$ are statistically dependent; same for $\mathbf{y}|\mathbf{z}$)
- We have developed an EM-based methodology to learn $p_{\mathbf{x}}$ and $p_{\mathbf{y}|\mathbf{z}}$ *online* and subsequently leverage this information for near-optimal Bayesian inference.

  J. P. Vila and P. Schniter, "Expectation-Maximization Gaussian-Mixture Approximate Message Passing," *IEEE Trans. Signal Process.*, Oct. 2013.
- We also have developed a "turbo" methodology that handles probabilistic dependencies among the elements of $\mathbf{x}$ and the elements of $\mathbf{y}|\mathbf{z}$.

  P. Schniter, "Turbo reconstruction of structured sparse signals," *Proc. CISS*, (Princeton, NJ), Mar. 2010.

Some Communications Applications of (EM/turbo) GAMP:

1. Communications over wideband channels

   - joint channel-estimation/equalization/decoding

     P. Schniter, "A Message-Passing Receiver for BICM-OFDM over Unknown Clustered-Sparse Channels," *IEEE J. Sel. Topics Signal Process.*, Dec. 2011.

     P. Schniter, "Belief-propagation-based joint channel estimation and decoding for spectrally efficient communication over unknown sparse channels," *Physical Communication*, Mar. 2012.

2. Communications over underwater channels

   - joint channel-tracking/equalization/decoding

     P. Schniter and D. Meng, "A Message-Passing Receiver for BICM-OFDM over Unknown Time-Varying Sparse Channels," *Allerton Conf.*, Sep. 2011.

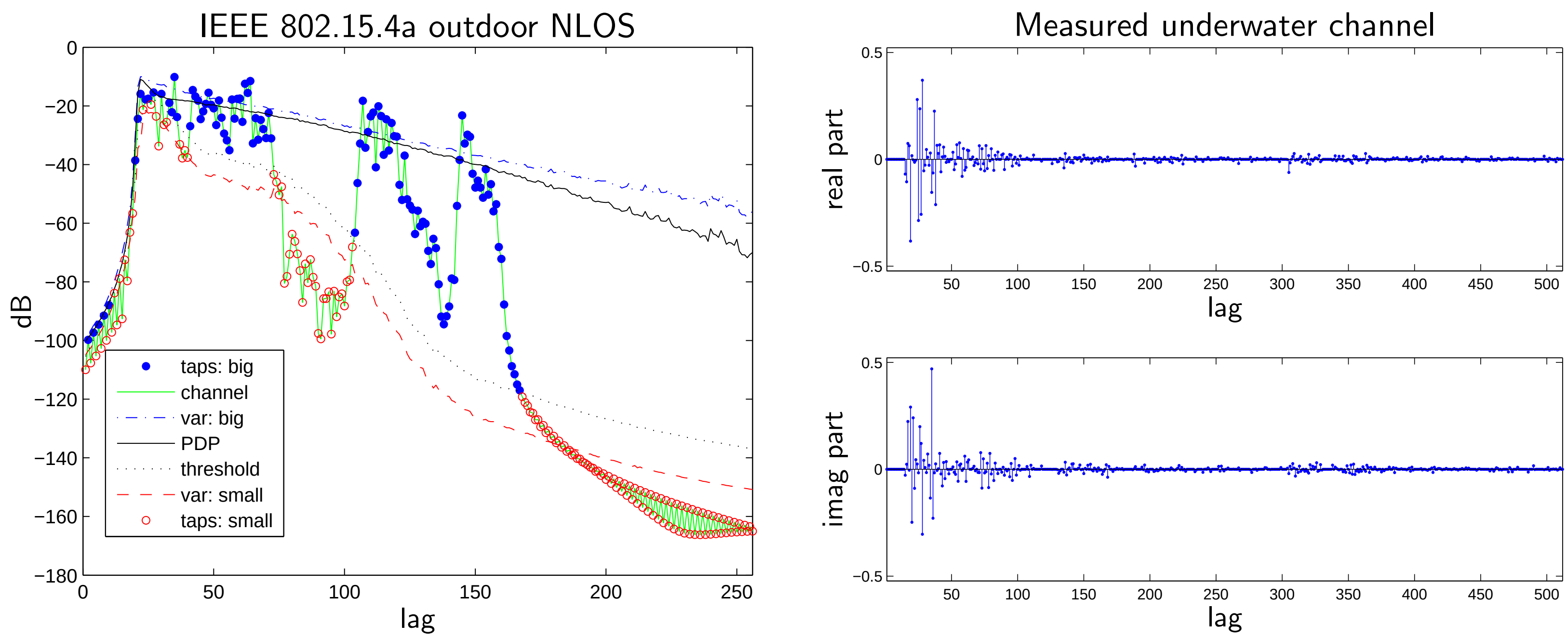
3. Communications in impulsive noise

   - joint channel-estimation/equalization/impulse-mitigation/decoding

     M. Nassar, P. Schniter, and B. Evans, "A Factor-Graph Approach to Joint OFDM Channel Estimation and Decoding in Impulsive Noise Environments," *IEEE Trans. Signal Process.*, to appear.

1. Comms over Wideband Channels:

- At large communication bandwidths, channel impulse responses are sparse.
- Below left shows channel taps $\boldsymbol{x} = [x_0, \ldots, x_{L-1}]$, where
  - $x_n = x(nT)$ for bandwidth $T^{-1} = 256$ MHz,
  - $x(t) = h(t) * p_{\mathsf{RC}}(t)$, and
  - $h(t)$ is generated randomly using 802.15.4a outdoor NLOS specs.



Simplified Channel Model:

First, let's simplify things to talk concretely about sparse channels. . .

Consider a discrete-time channel that is

- block-fading with block size N,
- frequency-selective with L taps (where $L < N$),
- sparse with S non-zero complex-Gaussian taps (where $0 < S \leq L$),

where *both the channel coefficients and support are unknown* to the receiver.

Important questions:

1. What is the capacity of this channel?
2. How can we build a practical comm system that operates near this capacity?

Noncoherent Capacity of the Sparse Channel:

For the unknown N-block-fading, L-length, S-sparse channel described earlier, we established that [1]

1. In the high-SNR regime, the ergodic capacity obeys

$$C_{\text{sparse}}(\text{SNR}) = \frac{N-S}{N}\log(\text{SNR}) + \mathcal{O}(1).$$

2. To achieve the prelog factor $R_{\text{sparse}} = \frac{N-S}{N}$, it suffices to use
   - pilot-aided OFDM (with N subcarriers, of which S are pilots)
   - with *joint* channel estimation and data decoding.

Key points:

- The effect of *unknown channel support* manifests only in the $\mathcal{O}(1)$ offset.
- [1] uses constructive proofs, but the decoder proposed there is not practical.

[1] A. Pachai-Kannu and P. Schniter, "On communication over unknown sparse frequency selective block-fading channels," *IEEE Trans. Info. Thy.*, Oct. 2011.

Practical Communication over the unknown Sparse Channel:

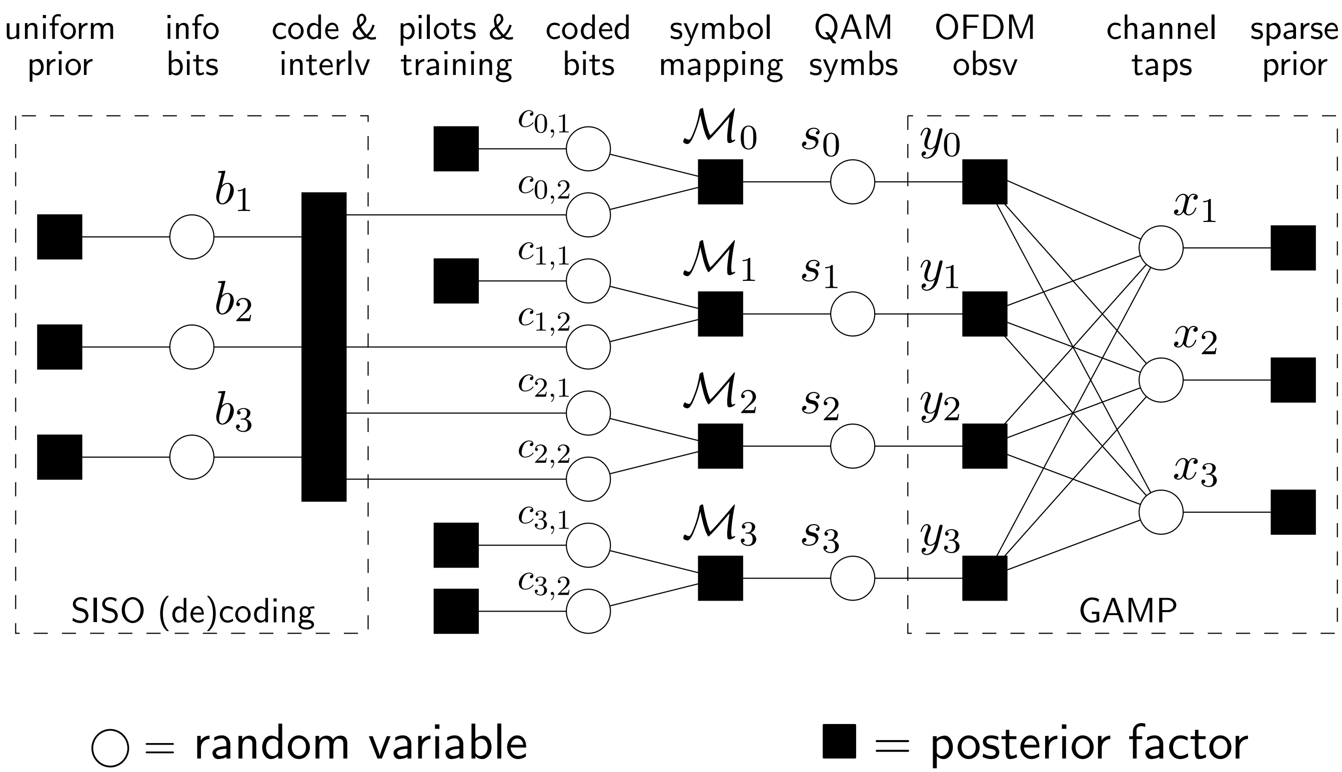
We now propose a communication scheme that. . .

- is practical, with decode complexity $\mathcal{O}(N \log_2 N + N|\mathbb{S}|)$ per block,
- (empirically) achieves the optimal prelog factor $R_{\text{sparse}} = \frac{N-S}{N}$,
- significantly outperforms "compressed channel sensing" (CCS) schemes.

Our scheme uses. . .

- a conventional transmitter: pilot-aided BICM OFDM,
- a novel receiver: based on GAMP.

Factor Graph for pilot-aided BICM-OFDM:



To jointly infer all random variables, we perform loopy-BP via the sum-product algorithm, using GAMP approximations in the GAMP sub-graph.

Numerical Results — Perfectly Sparse Channel:

Transmitter:

- LDPC codewords with length ~ 10000 bits.
- 2^M-QAM with $2^M \in \{4, 16, 64, 256\}$ and multi-level Gray mapping.
- OFDM with $N = 1024$ subcarriers.
- P pilot subcarriers and/or T training MSBs.

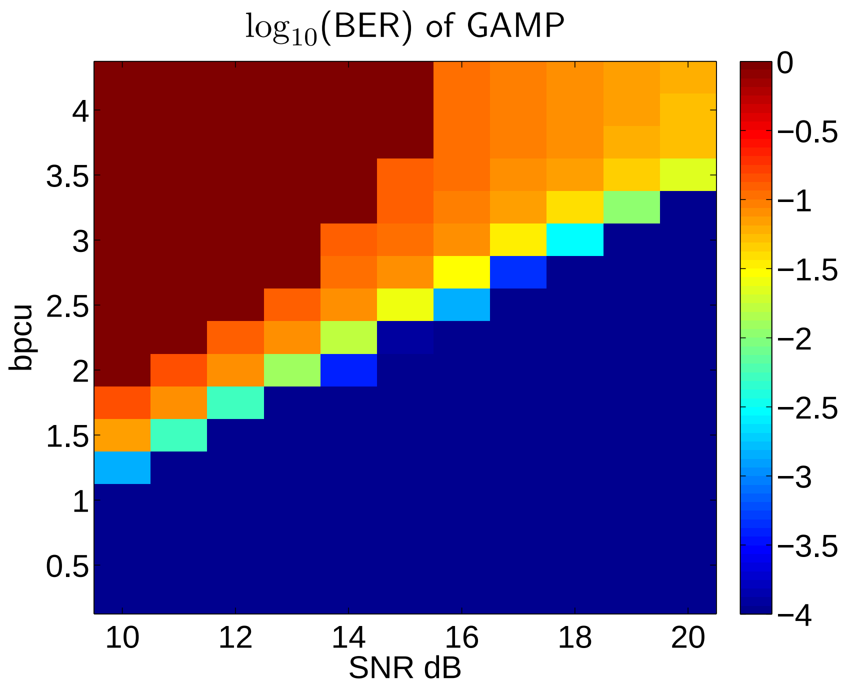
Channel:

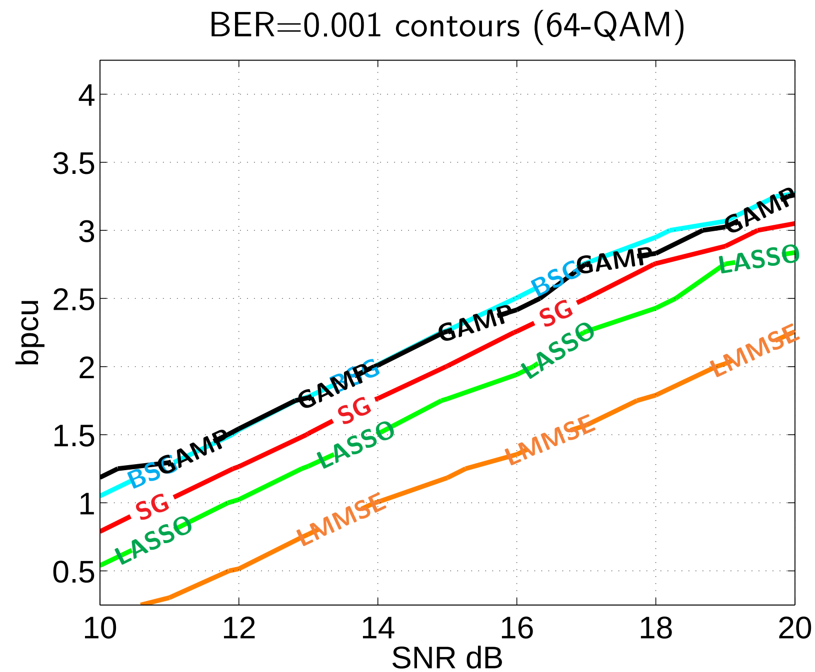
- Length $L = 256 = N/4$.
- Sparsity $S = 64 = L/4$.

Reference Schemes:

- Pilot-aided LASSO was implemented using SPGL1 with genie-aided tuning.
- Pilot-aided LMMSE, support-aware MMSE, and info-bit+support-aware MMSE channel estimates were also tested.

BER & Outage vs SNR (with $P = L$ pilots and $T = 0$ MSBs):

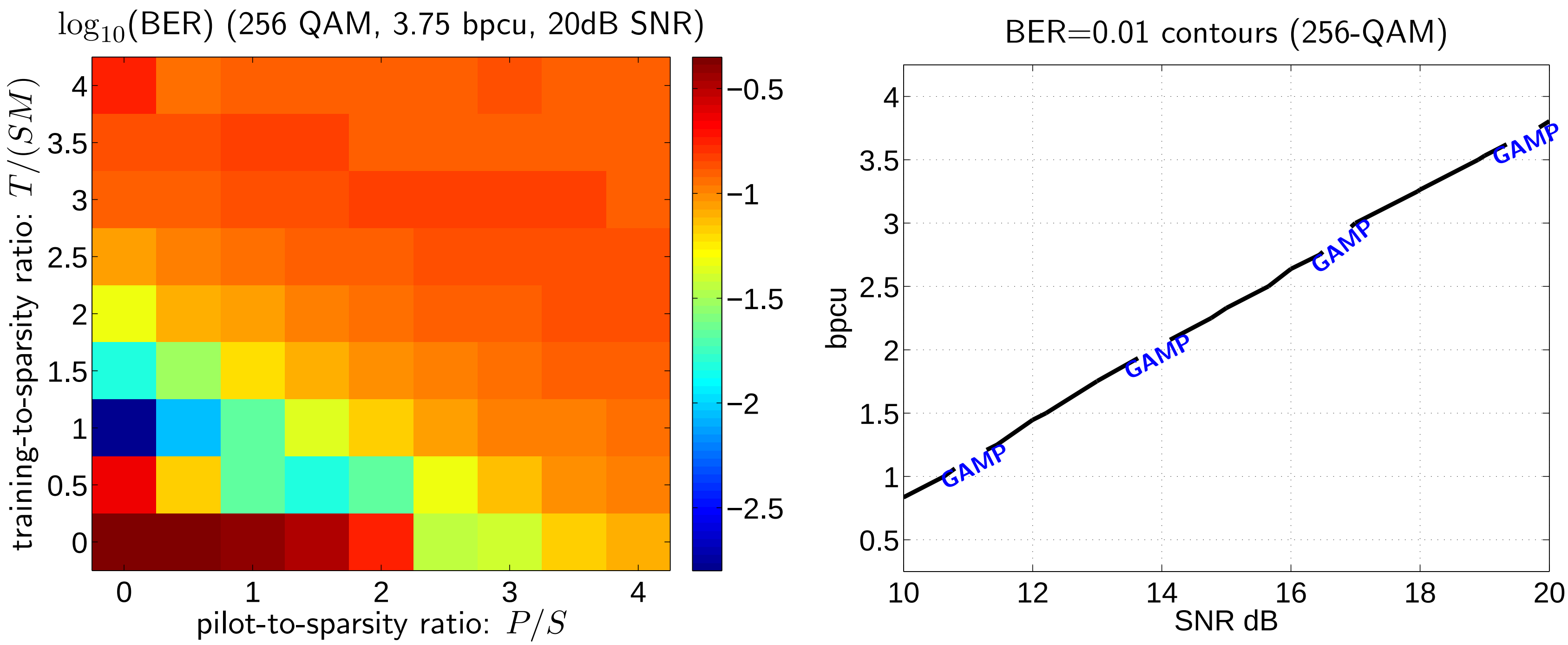




Key points:

- GAMP outperforms both LASSO and the support genie (SG).
- GAMP performs nearly as well as the info-bit+support-aware genie (BSG).
- With $P = L$, all approaches yield prelog factor $R = \frac{N-L}{N} = \frac{3}{4}$, which falls short of the optimal $R_{\text{sparse}} = \frac{N-S}{N} = \frac{15}{16}$.

BER & Outage vs SNR (with $P=0$ pilots & $T=SM$ training MSBs):

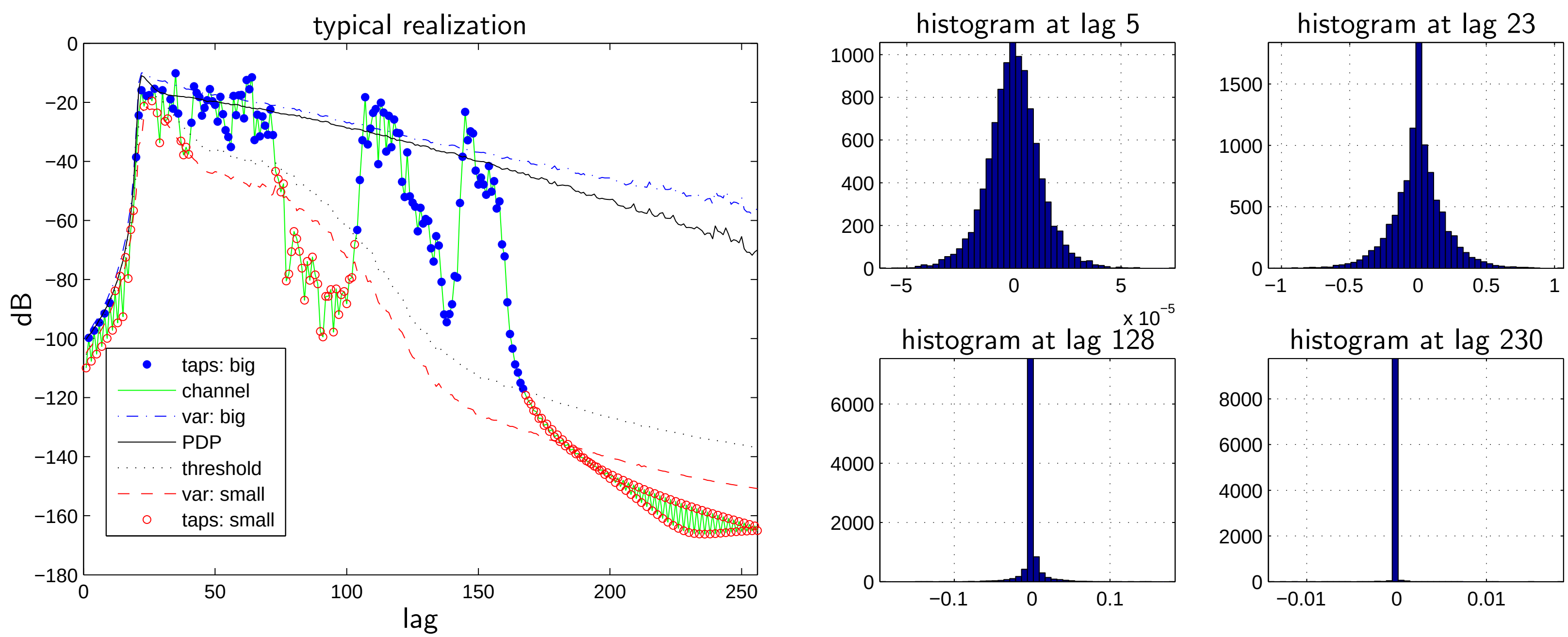


Key points:

- GAMP favors $P=0$ pilot subcarriers and $T=SM$ training MSBs.
  - Precisely the necc/suff redundancy of the capacity-maximizing system!
- GAMP achieves the sparse-channel's capacity-prelog factor, $R_{\text{sparse}} = \frac{N-S}{N}$.

In reality, channel taps are not perfectly sparse, nor i.i.d:

- For example, consider channel taps $\boldsymbol{x} = [x_0, \ldots, x_{L-1}]$, where
  - $x_n = x(nT)$ for bandwidth $T^{-1} = 256$ MHz,
  - $x(t) = h(t) * p_{\mathsf{RC}}(t)$, and
  - $h(t)$ is generated randomly using 802.15.4a outdoor NLOS specs.



- The tap distribution *varies as the lag increases*, becoming more heavy-tailed.
- The big taps are *clustered together* in lag, as are the small ones.

Proposed channel model:

- Saleh-Valenzuela (e.g., 802.15.4a) models are accurate but difficult to exploit in receiver design.
- We propose a structured-sparse channel model based on a 2-state Gaussian Mixture model with discrete-Markov-chain structure on the state:

$$p(x_j \,|\, d_j) \;=\; \begin{cases} \mathcal{CN}(x_j; 0, \mu_j^0) & \text{if } d_j = 0 \quad \text{“small”} \\ \mathcal{CN}(x_j; 0, \mu_j^1) & \text{if } d_j = 1 \quad \text{“big”} \end{cases}$$

$$\Pr\{d_{j+1} = 1\} \;=\; p_j^{10} \Pr\{d_j = 0\} + (1 - p_j^{01}) \Pr\{d_j = 1\}$$

- Our model is parameterized by the lag-dependent quantities:

$\{\mu_j^1\}$: big-state power-delay profile

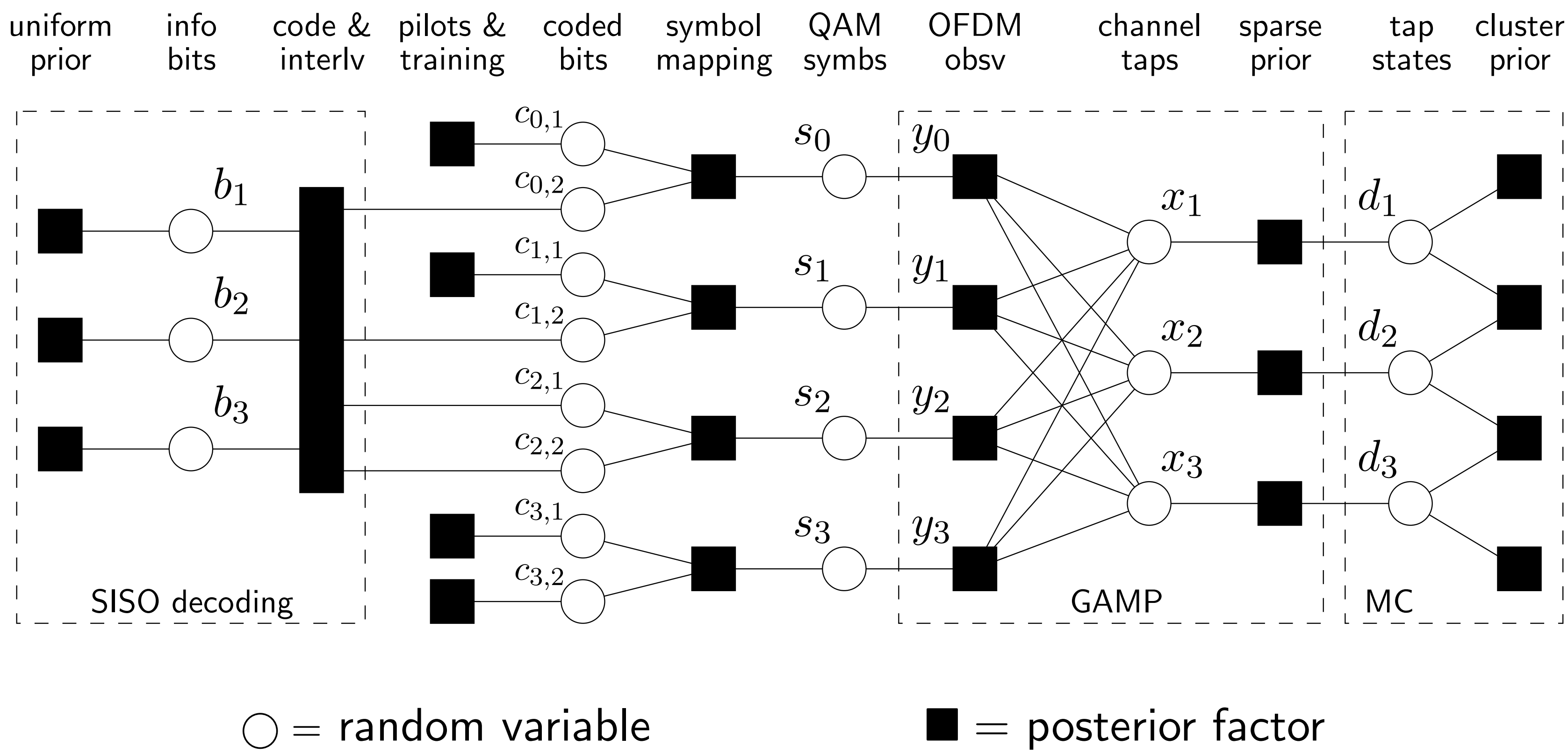
$\{\mu_j^0\}$: small-state power-delay profile

$\{p_j^{01}\}$: big-to-small transition probabilities

$\{p_j^{10}\}$: small-to-big transition probabilities

- Can learn these statistical params from observed realizations via the EM alg.

Factor graph for pilot-aided BICM-OFDM:



To jointly infer all random variables, we perform loopy-BP via the sum-product algorithm, using GAMP approximations in the GAMP sub-graph.

Numerical results:

Transmitter:

- OFDM with $N = 1024$ subcarriers.
- 16-QAM with multi-level Gray mapping
- LDPC codewords with length ~ 10000 yielding spectral efficiency of 2 bpcu.
- P "*pilot subcarriers*" and T "*training MSBs*."

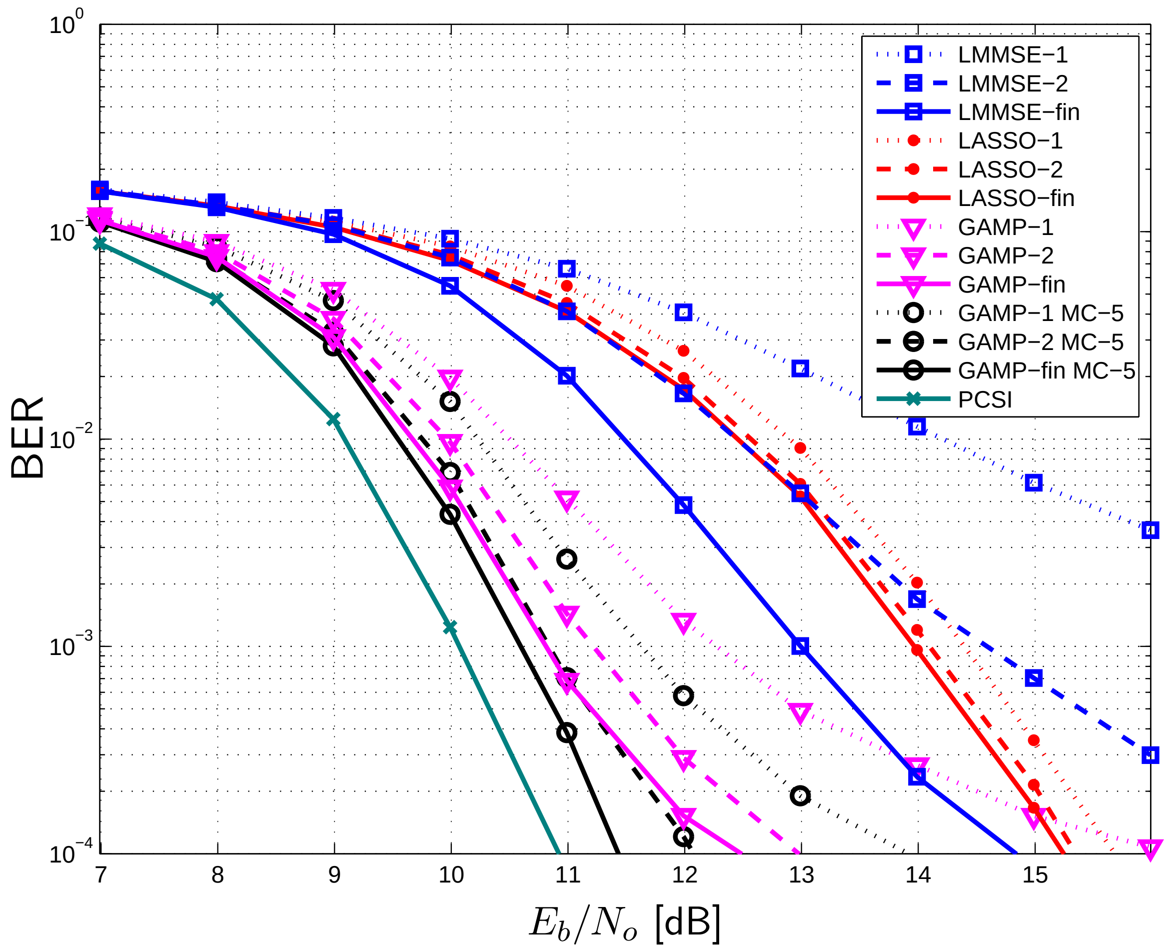
Channel:

- 802.15.4a outdoor-NLOS (not our Gaussian-mixture model!)
- Length $L = 256 = N/4$.

Reference Channel Estimation / Equalization Schemes:

- soft-input soft-output (SISO) versions of LMMSE and LASSO.
- perfect-CSI genie.

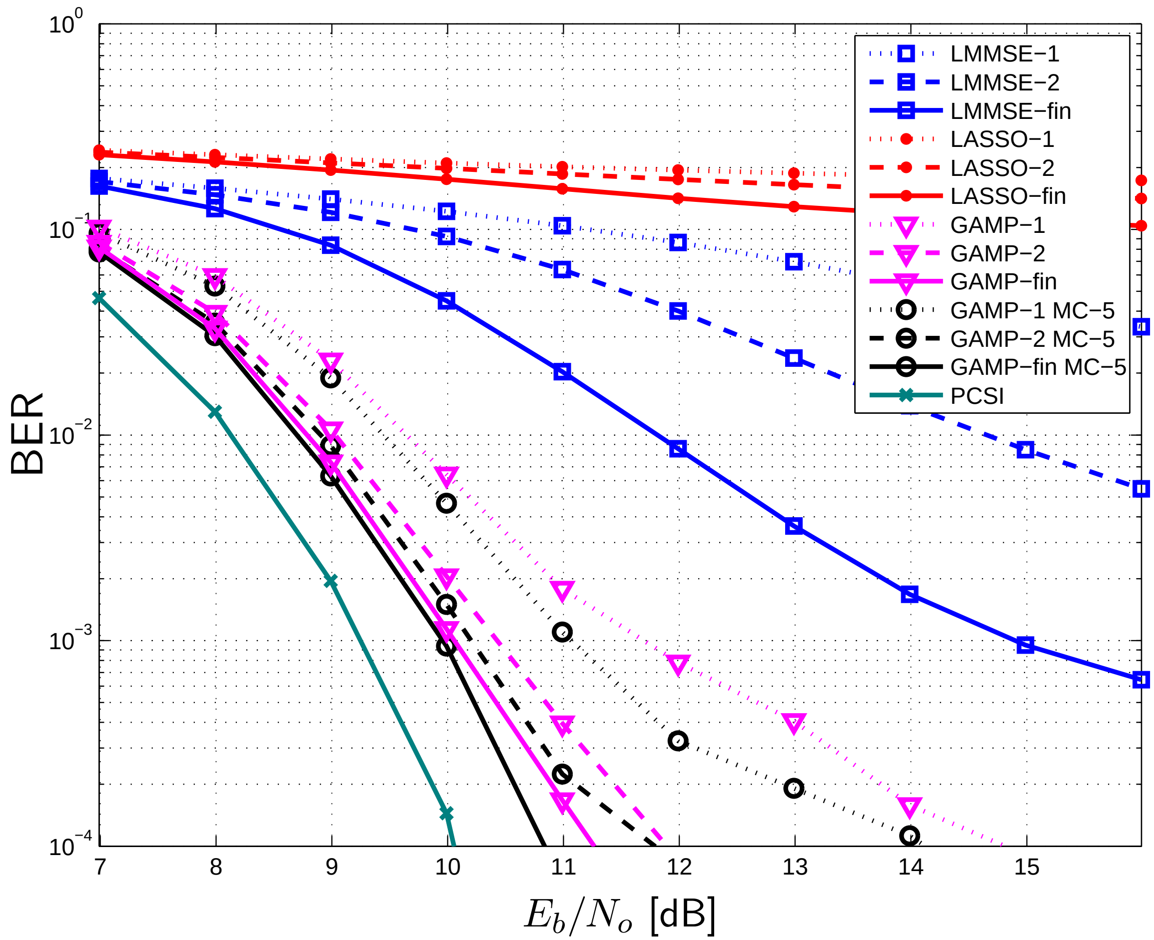
BER versus E_b/N_o for $P = 224$ pilots and $T = 0$ training MSBs:



Our scheme shows 4dB improvement over (turbo) LASSO.

Our scheme only 0.5dB from perfect-CSI genie!

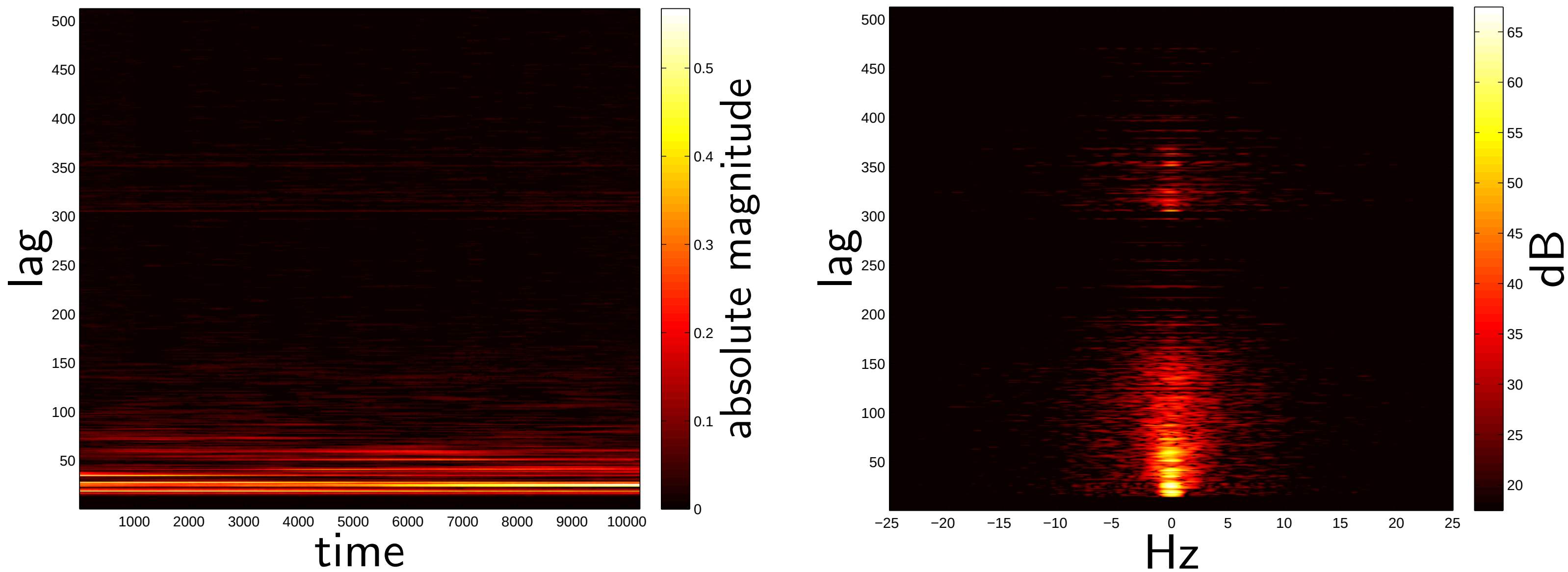
BER versus E_b/N_o for $P = 0$ pilots and $T = 448$ training MSBs:



Use of training MSBs gives 1dB improvement over use of pilot subcarriers!

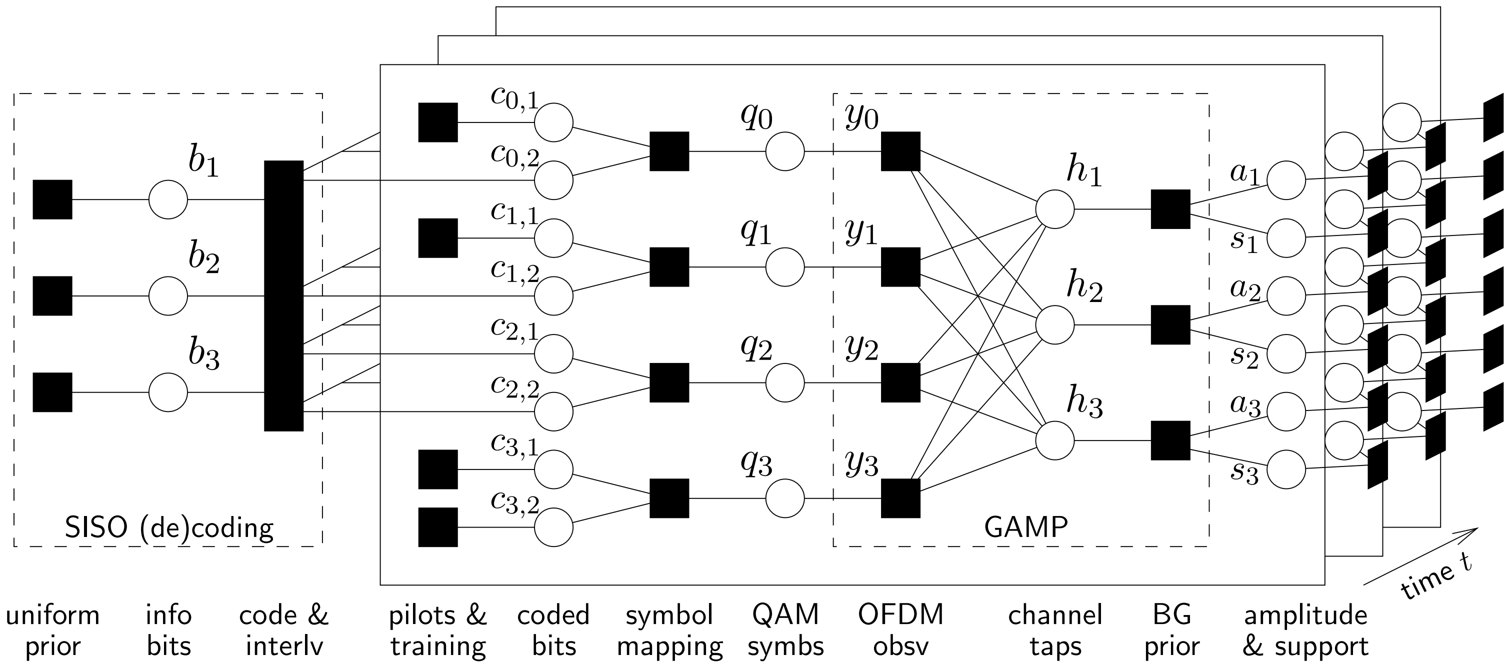
2. Communications over Underwater Channels:

- SPACE-08 Underwater Experiment `2920156F038_C0_S6`
- Time-varying channel response estimated using WHOI M-sequence:



- The channel is nearly over-spread: $f_d T_s L = 20 \times \frac{1}{10000} \times 400 = 0.8$!
- Can't afford to ignore structure of temporal variations!

BICM-OFDM Factor Graph with Temporal Channel Structure:

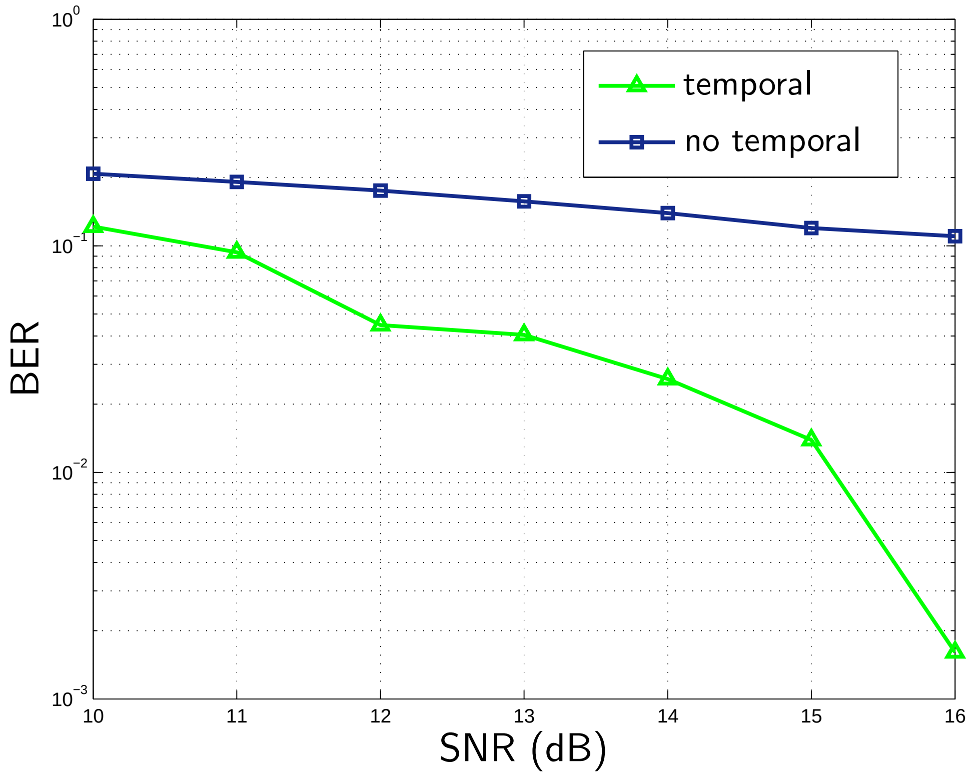


- Channel taps are modeled as independent Bernoulli-Gaussian processes:
  - each tap's amplitude follows a temporal Gauss-Markov chain
  - each tap's on/off state follows a temporal discrete-Markov chain

Performance versus SNR:

Settings:

- experimentally measured underwater channel
- 16-QAM
- 1024 total tones
- 0 pilot tones
- 256 training MSBs
- LDPC length 10k
- LDPC rate 0.5



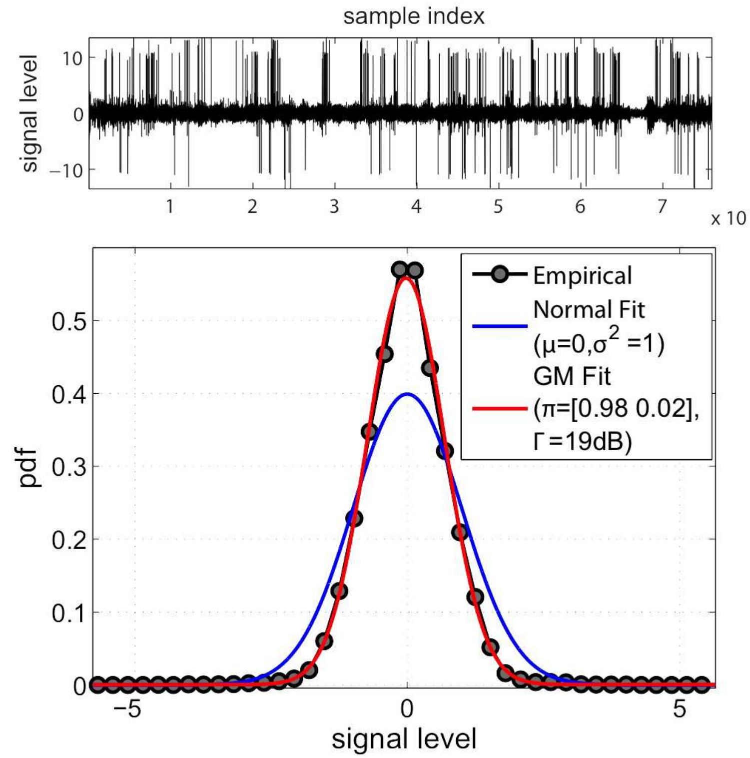
Exploiting the persistence in channel support and channel amplitudes was critical in this difficult underwater application.

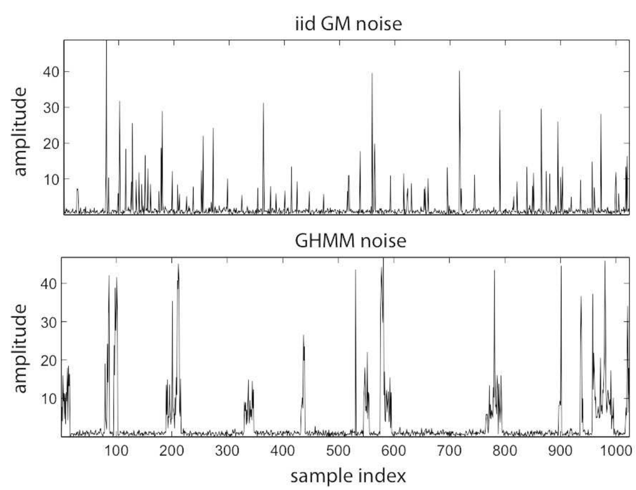
3. Communications in Impulsive Noise:

- In many wireless and power-line communication systems, the (time-domain) noise is not Gaussian but impulsive.

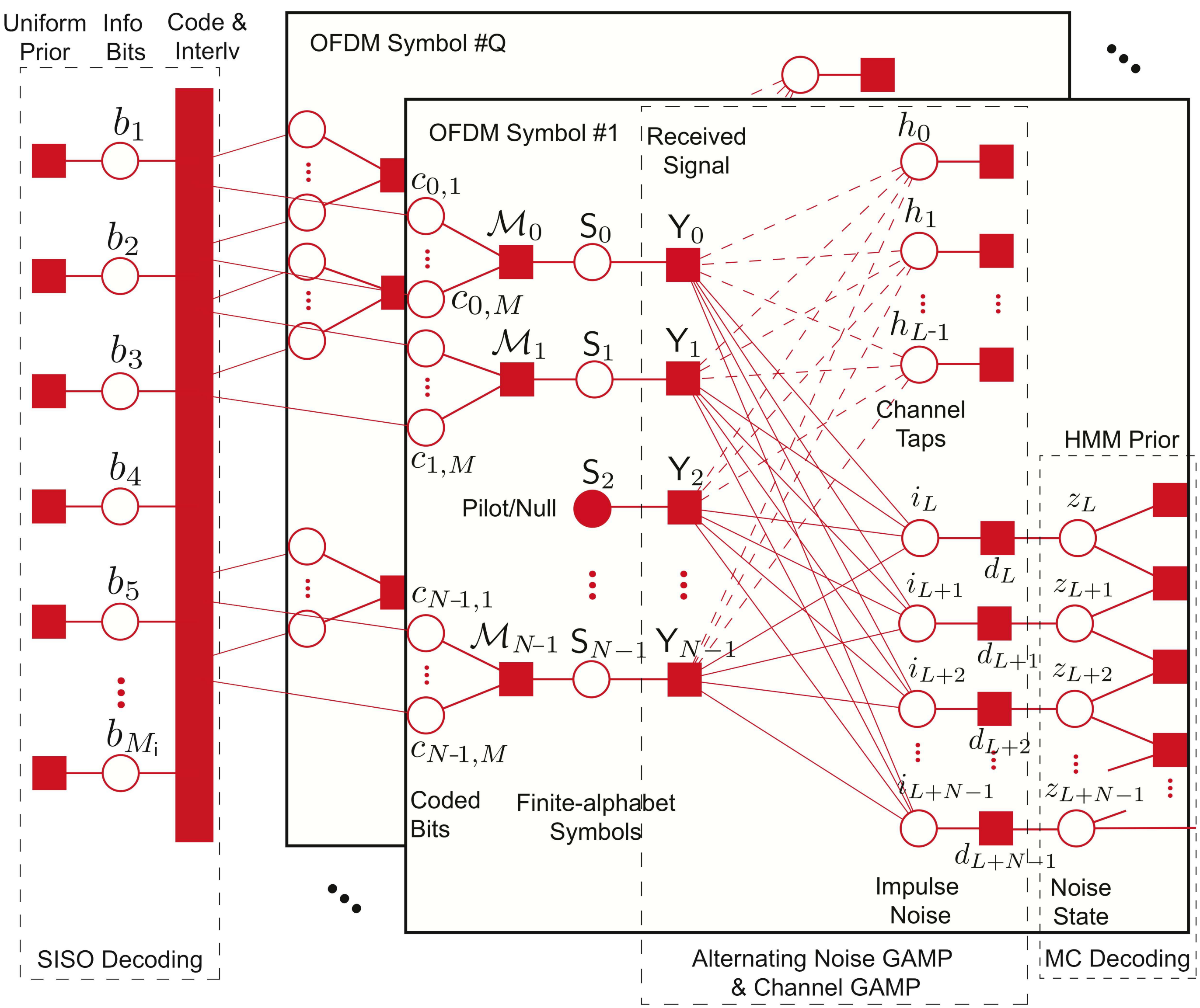
- The marginal noise statistics are well captured by a 2-state Gaussian mixture (i.e., Middleton class-A) model.

- Noise burstiness is well captured by a discrete Markov chain on the noise state.





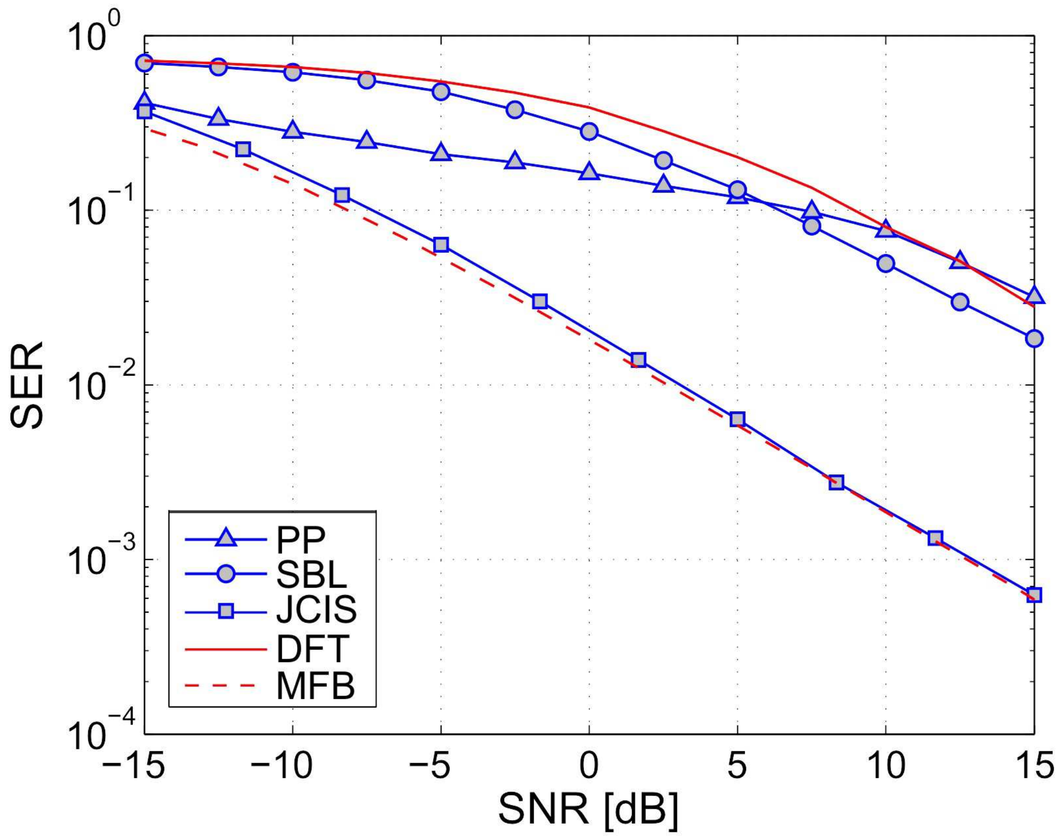
Factor Graph for pilot-aided BICM-OFDM:



Numerical Results — Uncoded Case:

Settings:

- 5 channel taps
- GM noise
- 256 total tones
- 15 pilot tones
- 80 null tones
- 4-QAM

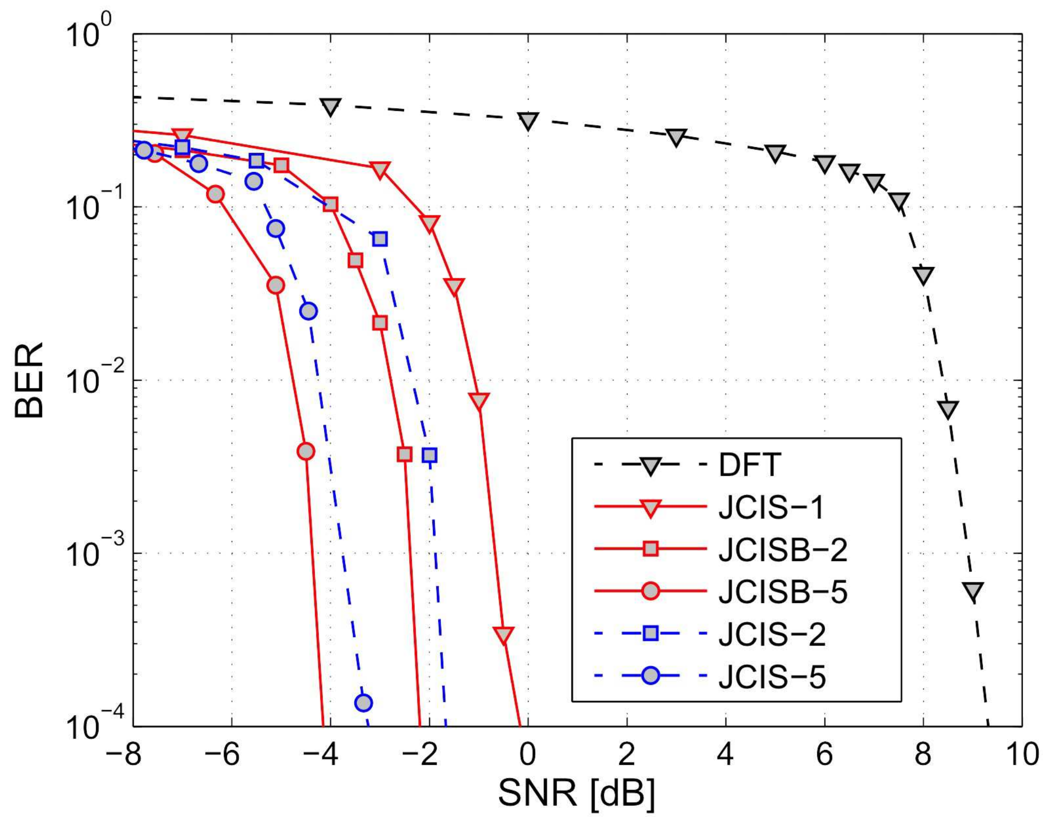


Proposed "joint channel/impulsive-noise/symbol" estimation (JCIS) scheme gives ∼15 dB gain over previous state-of-the-art and performs within 1 dB of MFB!

Numerical Results — Coded Case:

Settings:

- 10 channel taps
- GM noise
- 1024 total tones
- 150 pilot tones
- 0 null tones
- 16-QAM
- LDPC
- Rate 0.5
- Length 60k



Proposed "joint channel/impulsive-noise/symbol/bit" estimation (JCISB) scheme gives ∼15 dB gain over traditional DFT-based receiver!

Conclusions:

- Inference in the generalized linear model yields an important but challenging class of problems.
- The generalized approximate message passing (GAMP) is a important new tool for solving such problems (under sufficiently large and dense transforms).
- Problems of this form manifest in BICM-OFDM comms receivers, where one wants to optimally decode bits in the presence of unknown channels, symbols, and noise.
- Often, the channel and noise processes have interesting statistical structures (e.g., sparsity, clustering, time-variation) and decoding performance can be dramatically improved when these structures are properly exploited.
- For such problems, GAMP can be "plugged into" the standard "turbo" receiver architecture to yield near-optimal performance with manageable complexity.