Max-Diversity Affine Precoding for the Noncoherent Doubly Dispersive Channel

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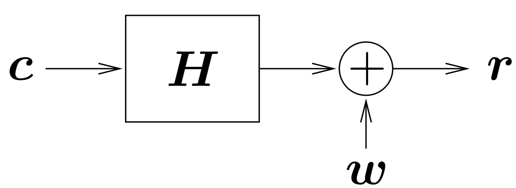
Motivating Questions:

1. What is the maximum achievable diversity order for communication over an *unknown* time/frequency-selective channel?
2. How should the transmitted signal be designed to facilitate maximum diversity reception?

System Model:

$$
\boxed{r_n \;=\; \sum_{l=0}^{N_h-1} h_{n,l}\, c_{n-l} + w_n}
$$

$$
\begin{aligned}
\boldsymbol{r} := [r_0, \ldots, r_{N-1}]^T &\quad:\quad \text{received samples}\\
\boldsymbol{c} := [c_0, \ldots, c_{N-1}]^T &\quad:\quad \text{coded symbols}\\
\boldsymbol{w} := [w_0, \ldots, w_{N-1}]^T &\quad:\quad \text{noise samples, } \mathcal{CN}(\mathbf{0}, \sigma^2 \boldsymbol{I})\\
\boldsymbol{H} &\quad:\quad \text{LTV channel matrix}
\end{aligned}
$$



$$
\boldsymbol{H} = \begin{bmatrix}
h_{0,0} & 0 & \cdots & 0 & \cdots & 0\\
\vdots & \ddots & \ddots & & \ddots & \vdots\\
h_{N_h-1,N_h-1} & \cdots & h_{N_h-1,0} & 0 & \ddots & 0\\
0 & h_{N_h,N_h-1} & \cdots & h_{N_h,0} & & 0\\
\vdots & \ddots & \ddots & & \ddots & \\
0 & \cdots & 0 & h_{N-1,N_h-1} & \cdots & h_{N-1,0}
\end{bmatrix}
$$

Karhunen-Loève Basis Expansion Model:

KL-BEM of l^{th}-tap trajectory over N-sample block:

$$\boldsymbol{h}_l := \begin{bmatrix} h_{0,l} \\ \vdots \\ h_{N-1,l} \end{bmatrix} = \boldsymbol{B}_l \boldsymbol{\theta}_l, \quad \boldsymbol{\theta}_l \in \mathbb{C}^{N_b}$$

N_b : Temporal degrees-of-freedom per tap

WSSUS Rayleigh channel assumption:

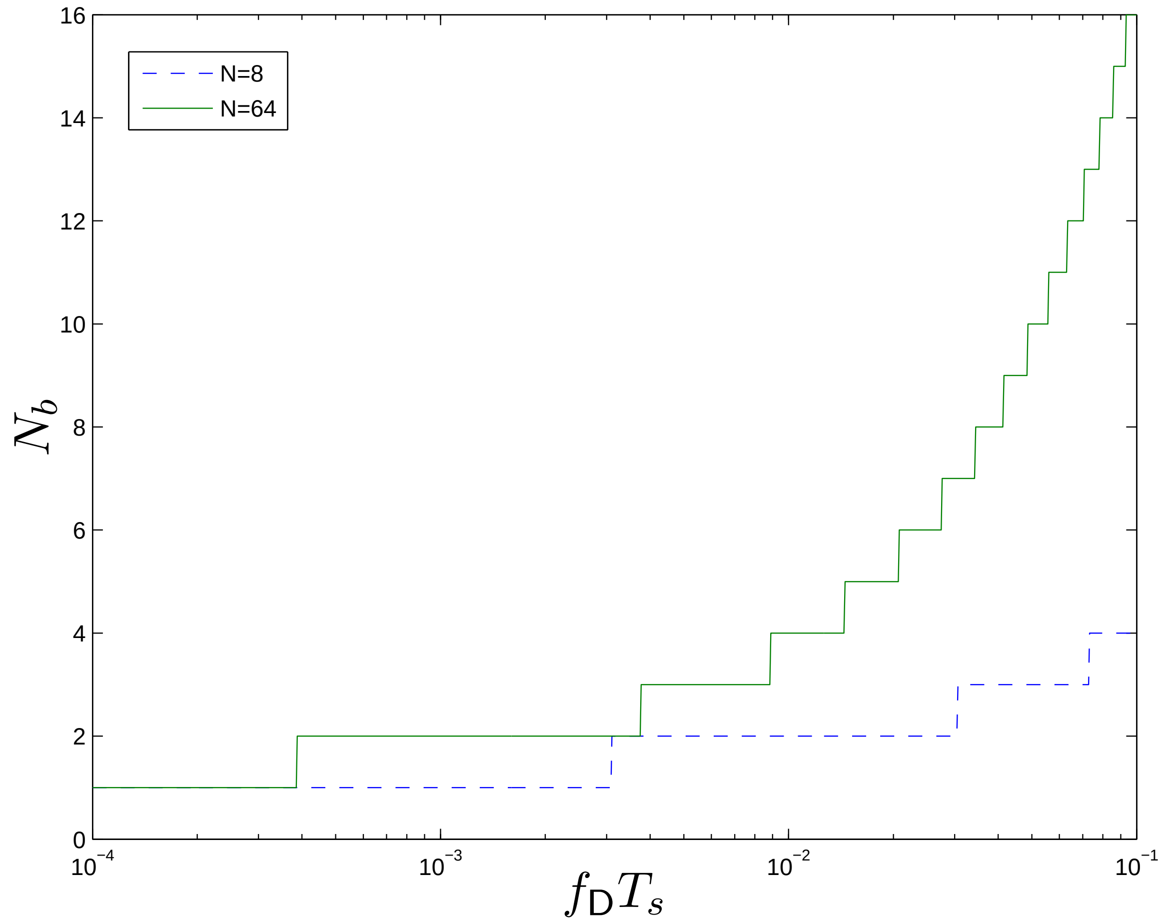
N_h : Number of taps

$$\boldsymbol{B} = \boldsymbol{B}_l \quad \forall l \in \{0, \ldots, N_h - 1\}$$

$$\boldsymbol{\theta} := \begin{bmatrix} \boldsymbol{\theta}_0 \\ \vdots \\ \boldsymbol{\theta}_{N_h-1} \end{bmatrix} \sim \mathcal{CN}(\mathbf{0}, \boldsymbol{R}_\theta), \quad \boldsymbol{R}_\theta \text{ has full rank } N_h N_b$$

Per-tap DoF N_b vs. normalized Doppler spread $f_D T_s$:

of eigenvalues within 30dB of largest eigenvalue (Jakes spectrum):



Notice that $N_b \ll N$.

Noncoherent ML Decoding:

Goal: Estimate $\boldsymbol{c} \in \boldsymbol{\mathcal{C}}$ from $\boldsymbol{r} = \boldsymbol{H}\boldsymbol{c} + \boldsymbol{w}$ assuming $\{\boldsymbol{H}, \boldsymbol{w}\}$ are unknown but statistics $\{\boldsymbol{B}, \boldsymbol{R}_\theta, \sigma^2\}$ are known.

Writing the received vector as

$$\boldsymbol{r} = \boldsymbol{C}_{\boldsymbol{B}}\boldsymbol{\theta} + \boldsymbol{w},$$

where matrix $\boldsymbol{C}_{\boldsymbol{B}}$ is composed of coded symbols $\boldsymbol{c}$ and basis vectors $\boldsymbol{B}$, the noncoherent ML estimate can be written

$$\begin{aligned}
\hat{\boldsymbol{c}}_{\mathsf{ML}} &= \arg\min_{\boldsymbol{c}\in\boldsymbol{\mathcal{C}}} \boldsymbol{r}^H \boldsymbol{\Phi} \boldsymbol{r} - \sigma^2 \log\det(\boldsymbol{C}_{\boldsymbol{B}}^H \boldsymbol{C}_{\boldsymbol{B}} + \sigma^2 \boldsymbol{R}_\theta^{-1}) \\
\boldsymbol{\mathcal{C}} &: \text{ set of code vectors} \\
\boldsymbol{\Phi} &:= \left(\boldsymbol{C}_{\boldsymbol{B}} \boldsymbol{R}_\theta \boldsymbol{C}_{\boldsymbol{B}}^H + \sigma^2 \boldsymbol{I}_N\right)^{-1}
\end{aligned}$$

Pair-Wise Error Probability:

Lemma 1 *Say* $\{\boldsymbol{C}_{\boldsymbol{B}}^{(k)}, \boldsymbol{C}_{\boldsymbol{B}}^{(l)}\}$ *are two possibilities for* $\boldsymbol{C}_{\boldsymbol{B}}$*. If the matrix*

$$\boldsymbol{M}_{kl} := \boldsymbol{C}_{\boldsymbol{B}}^{(k)H}\big(\boldsymbol{I}_N - \boldsymbol{C}_{\boldsymbol{B}}^{(l)}(\boldsymbol{C}_{\boldsymbol{B}}^{(l)H}\boldsymbol{C}_{\boldsymbol{B}}^{(l)})^{-1}\boldsymbol{C}_{\boldsymbol{B}}^{(l)H}\big)\boldsymbol{C}_{\boldsymbol{B}}^{(k)}$$

is full rank, then, at high SNR,

$$\mathit{PWEP}_{kl} = \left(\frac{1}{\sigma^2}\right)^{-N_h N_b} \det(\boldsymbol{R}_\theta \boldsymbol{M}_{kl})^{-1}\binom{2N_h N_b - 1}{N_h N_b}.$$

Furthermore, $\boldsymbol{M}_{kl}$ *has full rank* $N_h N_b$ *if and only if* $[\boldsymbol{C}_{\boldsymbol{B}}^{(k)}, (\boldsymbol{C}_{\boldsymbol{B}}^{(l)} - \boldsymbol{C}_{\boldsymbol{B}}^{(k)})]$ *has full rank* $2N_h N_b$*.*

Main points:

1. $N_h N_b$ is the maximum achievable diversity order.
2. Max-diversity requires full-rank $[\boldsymbol{C}_{\boldsymbol{B}}^{(k)}, (\boldsymbol{C}_{\boldsymbol{B}}^{(l)} - \boldsymbol{C}_{\boldsymbol{B}}^{(k)})]$ $\forall k \neq l$ which requires $N \geq 2N_h N_b$.

Linear Precoding:

Say that the code vectors are generated via

$$\boxed{\boldsymbol{c} \;=\; \boldsymbol{P}\boldsymbol{s}} \qquad \boldsymbol{s} \in \boldsymbol{\mathcal{S}} \subset \mathbb{C}^{N_s}$$

where $\boldsymbol{P} \in \mathbb{C}^{N \times N_s}$ is a linear precoding matrix and $\boldsymbol{s}$ is a symbol vector.

Lemma 2 *Linear precoding does not facilitate maximum-diversity decoding whenever the symbol vector alphabet $\boldsymbol{\mathcal{S}}$ contains elements that differ by no more than a scale factor (e.g., uncoded QAM or PSK).*

Affine Precoding:

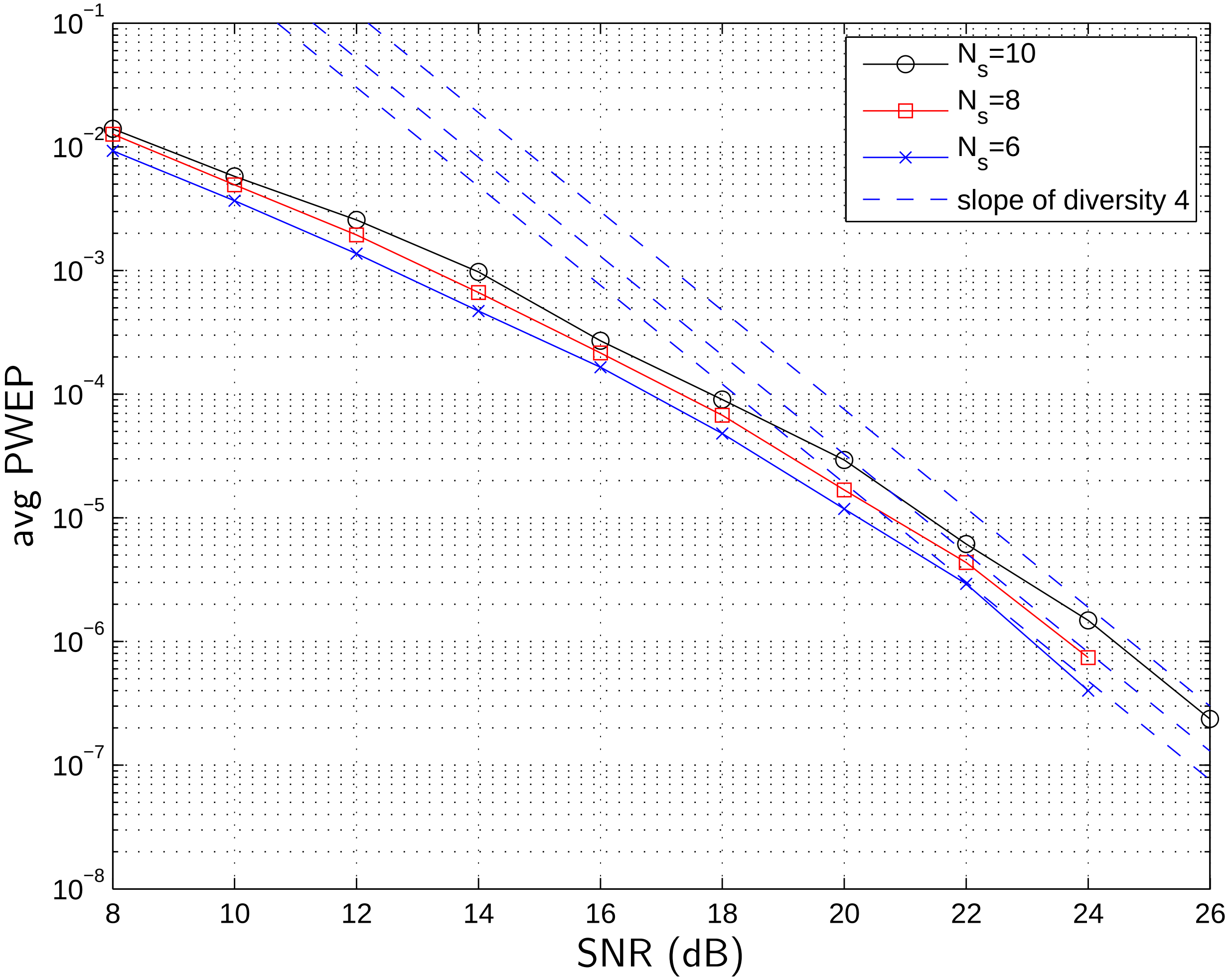
$$\boxed{\boldsymbol{c} = \boldsymbol{P}\boldsymbol{s} + \boldsymbol{t}} \qquad \boldsymbol{s} \in \boldsymbol{\mathcal{S}} \subset \mathbb{C}^{N_s}.$$

Lemma 3 *If $N \geq 2N_h N_b$ and if the matrix created from the last $N - N_h + 1$ rows of $\boldsymbol{B}$ is full rank, then choosing $[\boldsymbol{P}, \boldsymbol{t}]$ randomly ensures that $[\boldsymbol{C}_{\boldsymbol{B}}^{(k)}, (\boldsymbol{C}_{\boldsymbol{B}}^{(l)} - \boldsymbol{C}_{\boldsymbol{B}}^{(k)})]$ is full-rank w.p.1.*

Main points:

1. *Almost any* affine precoder provides maximum diversity!
2. There are no restrictions on the data rate N_s/N!
3. The rank condition on $\boldsymbol{B}$ is mild. (It requires that the first $N_h - 1$ samples of the N-sample tap trajectory are non-essential to experiencing the N_b degrees-of-freedom.)

Numerical Example:



$N = 8$

$N_h = 2$

$N_b = 2$

Jakes fading

$\boldsymbol{\mathcal{S}} = \text{BPSK}$

Systematic Affine Precoding:

$$\boxed{\boldsymbol{c} = \boldsymbol{P}\boldsymbol{s} + \boldsymbol{t}} \text{ with } \boldsymbol{P} = \begin{bmatrix} \boldsymbol{I}_{N_s} \\ \boldsymbol{P}' \end{bmatrix}, \quad \boldsymbol{P}' \in \mathbb{C}^{N_p \times N_s}$$

Lemma 4 *If $N_p \geq N_h N_b - 1$, if $N \geq 2N_h N_b$, and if matrix created from the last $N_p - N_h + 1$ rows of $\boldsymbol{B}$ is full-rank, then choosing $[\boldsymbol{P}', \boldsymbol{t}]$ randomly ensures that $[\boldsymbol{C}_{\boldsymbol{B}}^{(k)}, (\boldsymbol{C}_{\boldsymbol{B}}^{(l)} - \boldsymbol{C}_{\boldsymbol{B}}^{(k)})]$ is full-rank w.p.1.*

Main points:

1. Systematic affine precoding facilitates *fast decoding*.
2. With $N_p \geq N_h N_b - 1$, almost any precoder provides max-diversity!
3. Rate limitation: $\frac{N_s}{N} \leq 1 - \frac{N_h N_b - 1}{N}$.
4. As before, the rank condition on $\boldsymbol{B}$ is mild.

Conclusions:

For noncoherent communication over a WSSUS time/frequency-selective channel with N_h delay taps and N_b temporal degrees-of-freedom per tap,

1. the maximum diversity order is $N_h N_b$,
2. block lengths $N \geq 2N_h N_b$ faciltate max-diversity reception,
3. linear precoding does *not* facilitate max-diversity reception,
4. *almost any* affine precoder facilitates max-diversity reception *at any rate* $\frac{N_s}{N}$,
5. *systematic* affine precoding facilitates max-diversity at rates $\frac{N_s}{N} \leq 1 - \frac{N_h N_b - 1}{N}$ while simplifying the decoding task.