Bilinear Generalized Approximate Message Passing (BiG-AMP) for Matrix Recovery Problems

Phil Schniter and Volkan Cevher

(With support from NSF CCF-1018368, DARPA/ONR N66001-10-1-4090, ERC Future Proof, MIRG-268398, SNR 200021-132548, DARPA KECoM, and a Rice Faculty Fellowship)

ITA — 2/9/2012

Three Important Matrix Recovery Problems:

- **Matrix Completion** (MC):

  Recover low-rank matrix $\boldsymbol{X}$ from AWGN-corrupted incomplete observations $\boldsymbol{Y} = \mathcal{P}_\Omega\big(\boldsymbol{X}+\boldsymbol{W}\big)$.

- **Robust Principle Components Analysis** (RPCA):

  Recover low-rank matrix $\boldsymbol{X}$ and sparse matrix $\boldsymbol{S}$ from AWGN-corrupted observations $\boldsymbol{Y} = \boldsymbol{X}+\boldsymbol{S}+\boldsymbol{W}$.

- **Dictionary Learning** (DL):

  Recover overcomplete dictionary $\boldsymbol{A}$ and sparse matrix $\boldsymbol{S}$ from AWGN-corrupted observations $\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{S}+\boldsymbol{W}$.

The following **extensions** may also be of interest:

- RPCA and DL with incomplete observations and/or structured sparsity.
- Any of the above with a non-additive noise model (e.g., quantized $\boldsymbol{Y}$).

Our contribution:

- We propose a novel unified approach to these matrix-recovery problems that leverages the recent framework of **approximate message passing** (AMP).

- While previous AMP algorithms have been proposed for the **linear model**:
  - Infer $\boldsymbol{s} \sim \prod_n p_S(s_n)$ from $\boldsymbol{y} = \boldsymbol{\Phi s} + \boldsymbol{w}$ with AWGN $\boldsymbol{w}$ and known $\boldsymbol{\Phi}$ [Donoho/Maleki/Montanari'10]

  or the **generalized linear model**:
  - Infer $\boldsymbol{s} \sim \prod_n p_S(s_n)$ from $\boldsymbol{y} \sim \prod_m p_{Y|X}(y_m|x_m)$ with hidden $\boldsymbol{x} = \boldsymbol{\Phi s}$ and known $\boldsymbol{\Phi}$ [Rangan'10]

  our new algorithm is formulated for the **generalized bilinear model**:
  - Infer $\boldsymbol{A} \sim \prod_{m,r} p_A(a_{mr})$ and $\boldsymbol{B} \sim \prod_{r,n} p_B(b_{rn})$ from $\boldsymbol{Y} \sim \prod_{m,n} p_{Y|X}(y_{mn}|x_{mn})$ with hidden $\boldsymbol{X} = \boldsymbol{AB}$ [Schniter/Cevher'11]

- Although our work is still **in-progress**, the preliminary results look very encouraging!

Outline:

1. **Brief review** of popular approaches to matrix-completion and robust PCA:
   - Convex
   - Greedy
   - Bayesian
2. **Bilinear Generalized AMP (BiG-AMP)**.
   - What is it?
   - What are AMP's approximations?
   - How to apply to MC, RPCA, DL?
3. **Preliminary results**:
   - Phase transition curves
   - NMSE and runtime
   - Practical example: video surveillance

Convex-Optimization for Matrix-Completion & Robust PCA:

- Consider the combined MC-and-RPCA problem:

  Recover low-rank $\boldsymbol{X}$ and sparse $\boldsymbol{S}$ from AWGN-corrupted incomplete observations $\boldsymbol{Y} = \mathcal{P}_\Omega(\boldsymbol{X} + \boldsymbol{S} + \boldsymbol{W})$.

- Optimization approach:

$$\min_{\boldsymbol{X},\boldsymbol{S}} \left\{ \operatorname{rank}(\boldsymbol{X}) + \gamma \|\boldsymbol{S}\|_0 \right\} \text{ s.t. } \|\mathcal{P}_\Omega(\boldsymbol{X}+\boldsymbol{S}) - \boldsymbol{Y}\|_F \leq \eta \quad \ldots \textbf{intractable}$$

$$\min_{\boldsymbol{X},\boldsymbol{S}} \left\{ \|\boldsymbol{X}\|_* + \gamma \|\boldsymbol{S}\|_1 \right\} \text{ s.t. } \|\mathcal{P}_\Omega(\boldsymbol{X}+\boldsymbol{S}) - \boldsymbol{Y}\|_F \leq \eta \quad \ldots \textbf{convex}!$$

- Convex relaxation yields **perfect noiseless** & **stable noisy** recovery when:
  - $\operatorname{rank}(\boldsymbol{X})$ is sufficiently small,
  - singular vectors of $\boldsymbol{X}$ are not too cross-correlated nor too spiky,
  - support of $\boldsymbol{S}$ is random and sufficiently sparse,
  - observation set Ω is random and sufficiently large.

  Details given in, e.g., [Candés/Recht'08], [Candés/Plan'09], [Candés/Li/Ma/Wright'09], [Zhou/Wright/Li/Candés/Ma'10], and [Chen/Jalali/Sanghavi/Caramanis'11].

Fast Algorithms for Convex Matrix-Completion & Robust PCA:

- A comparison of convex RPCA algorithms is given at Yi Ma's webpage:

  `http://perception.csl.uiuc.edu/matrix-rank/sample_code.html`

Algorithm	Error	Time (sec)
Singular Value Thresholding [Cai/Candes/Shen'08]	3.4e-4	**877**
Dual Method [Lin/Ganesh/Wright/Wu/Chen/Ma'09]	1.6e-5	**177**
Accelerated Proximal Gradient (partial SVD) [Lin/Ganesh/Wright/Wu/Chen/Ma'09]	1.8e-5	**8**
Alternating Direction Methods [Yuan/Yang'09]	2.2e-5	**5**
Exact Augmented Lagrange Method [Lin/Chen/Wu/Ma'09]	7.6e-8	**4**
Inexact Augmented Lagrange Method [Lin/Chen/Wu/Ma'09]	4.3e-8	**2**

  for the recovery of 400×400 rank-20 matrix $\boldsymbol{X}$ corrupted by 5%-sparse $\boldsymbol{S}$ with amplitudes uniform in $[-50, 50]$.

- Evidently a lot of progress has been made! Can one do better?

Greedy Approaches to Matrix-Completion & Robust PCA:

- First consider **matrix completion**, where we want to recover low-rank $\boldsymbol{X}$ from AWGN-corrupted incomplete observations $\boldsymbol{Y} = \mathcal{P}_\Omega(\boldsymbol{X}+\boldsymbol{W})$.
- If we suppose that ...

  $\boldsymbol{X} \in \mathbb{R}^{M\times N}$ is square or tall (i.e., $M \geq N$) with $\operatorname{rank}(\boldsymbol{X}) = R$,

  then the difficult part of the MC problem is finding the column space of $\boldsymbol{X}$, leading to squared-error minimization on the **Grassmanian manifold** $\mathcal{G}_{M,R}$:

  $$\min_{\boldsymbol{A}\in\mathcal{G}_{M,R}} \min_{\boldsymbol{B}} \|\mathcal{P}_\Omega(\boldsymbol{A}\boldsymbol{B}) - \boldsymbol{Y}\|_F^2$$

- Example algorithms:
  - **Optspace** [Keshavan/Montanari/Oh'09]: Grad-descent minimizing $(\boldsymbol{A},\boldsymbol{B})$.
  - **SET** [Dai/Milenkovic'09]: Solves for $\boldsymbol{B}$, then takes gradient w.r.t $\boldsymbol{A}$.
  - **GROUSE** [Balzano/Nowak/Recht'10]: Grad-descent one column at a time.
- This greedy approach can also be extended to **RPCA**:
  - **GRASTA** [He/Balzano/Lui'11].

Bayesian Approaches to Matrix-Completion & Robust PCA:

- First consider **matrix completion**, where we want to recover low-rank $\boldsymbol{X}$ from AWGN-corrupted incomplete observations $\boldsymbol{Y} = \mathcal{P}_\Omega(\boldsymbol{X} + \boldsymbol{W})$.
- The *basic* Bayesian approach decomposes $\boldsymbol{X} = \boldsymbol{AB}$ and assumes priors $\boldsymbol{A} \sim \mathcal{N}(\boldsymbol{0}, \sigma_A^2 \boldsymbol{I})$ and $\boldsymbol{B} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{I})$. The log posterior then becomes

$$\ln p(\boldsymbol{A}, \boldsymbol{B}|\boldsymbol{Y}) = \tfrac{1}{2\sigma_W^2} \|\mathcal{P}_\Omega(\boldsymbol{AB}) - \boldsymbol{Y}\|_F^2 + \tfrac{1}{2\sigma_A^2}\|\boldsymbol{A}\|_F^2 + \tfrac{1}{2}\|\boldsymbol{B}\|_F^2 + C.$$

  To infer $(\boldsymbol{A}, \boldsymbol{B})$, various schemes have been proposed, e.g.,

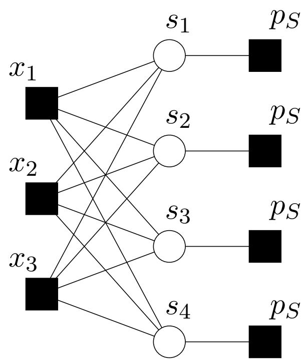
  - **EM** ("Probabilistic PCA") [Tipping/Bishop'99]
  - **SDP** ("Maximum-Margin Matrix Factorization") [Srebro/Rennie/Jaakkola'04]
  - **VB** ("Variational Bayes") [Lim/Teh'07]
  - **MCMC** ("Probabilistic Matrix Factorization") [Salakhutdinov/Mnih'08]

  Each has their own way of estimating the hyperparameters $\{\sigma_W^2, \sigma_A^2\}$.
- This approach can be extended to **RPCA** by changing the noise model to a heavy-tailed one (e.g., [Luttinen/Ilin/Karhunen'09], [Ding/He/Carin'11]).

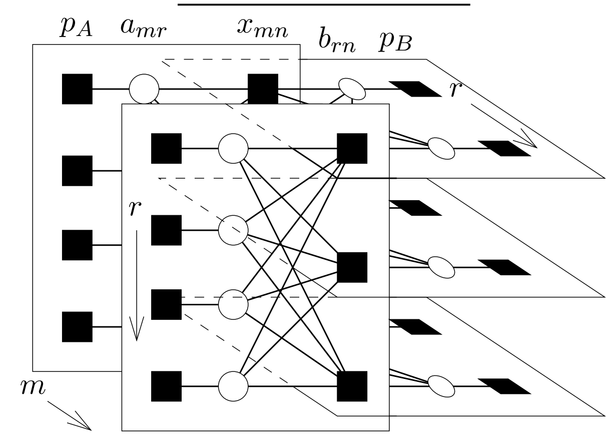
Bilinear Generalized AMP (BiG-AMP):

- **BiG-AMP** is a Bayesian approach that uses **approximate message passing (AMP)** strategies to infer $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{S})$.

Compressive Sensing (CS):

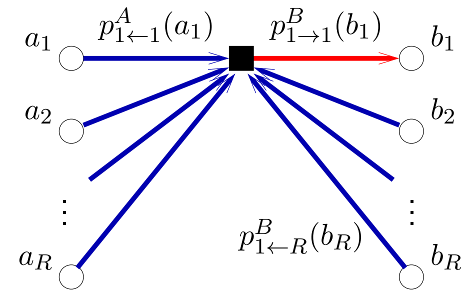


MC/RPCA/DL:



- In AMP, beliefs are propagated on a loopy factor graph using approximations that exploit the **blessings of dimensionality**:
  1. Gaussian message approximation (motivated by CLT),
  2. Taylor-series approximation of message differences.
- A rigorous large-system analysis of AMP for CS (with i.i.d Gaussian $\boldsymbol{\Phi}$) has established a number of optimalities [Bayati/Montanari'10],[Rangan'10].

BiG-AMP Approximations (sum-product version):



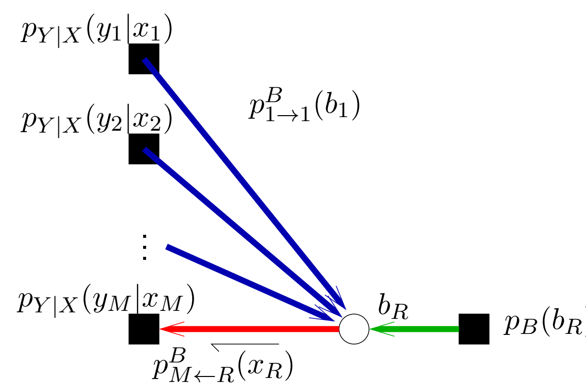
1. Message from i^{th} node of $\boldsymbol{X}$ to j^{th} node of $\boldsymbol{B}$:

$$p^B_{i\to j}(b_j) \propto \int_{\{a_r\}_{r=1}^R,\{b_r\}_{r\neq j}} p_{Y|X}\Big(y_i\Big|\overbrace{\textstyle\sum_r a_r b_r}^{x_i|b_j \approx \mathcal{N} \text{ via CLT!}}\Big)\Big(\textstyle\prod_r p^B_{i\leftarrow r}(b_r)\Big)\Big(\textstyle\prod_{r\neq j} p^A_{i\leftarrow r}(a_r)\Big)$$

$$\approx \int_{x_i} p_{Y|X}(y_i|x_i)\,\mathcal{N}\big(x_i;\hat{x}_i(b_j),\nu^x_i(b_j)\big) \approx \mathcal{N} \text{ (exact for AWGN!)}$$

The means and variances of $p^B_{i\leftarrow r}, p^A_{i\leftarrow r}$ suffice to compute $\hat{x}_i(b_j), \nu^x_i(b_j)$, thus **Gaussian message passing!** (Same thing happens with $\boldsymbol{X}\to\boldsymbol{A}$ messages.)

2. Although Gaussian, we still have $4MNR$ messages to compute (too many!). Exploiting similarity among the messages $\{p^B_{i\leftarrow j}\}_{i=1}^M$, AMP employs a Taylor-series approximation whose error vanishes as $M\to\infty$. (Same for $\{p^A_{i\leftarrow j}\}_{i=1}^N$.) In the end, AMP only needs to compute $\mathcal{O}(MN)$ **messages**!



BiG-AMP for MC, RPCA, and DL:

BiG-AMP can be applied to a wide variety of matrix recovery problems:

- **Matrix Completion** (MC):

  Recover low-rank $\boldsymbol{AB}$ from $\boldsymbol{Y} = \mathcal{P}_\Omega(\boldsymbol{AB} + \boldsymbol{W})$.

  ... set $\boldsymbol{A} \sim \mathcal{N}(\boldsymbol{0}, \sigma_A^2 \boldsymbol{I})$ and $\boldsymbol{B} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{I})$.

- **Robust PCA** (RPCA):

  Recover low-rank $\boldsymbol{AB}$ and sparse $\boldsymbol{S}$ from $\boldsymbol{Y} = \boldsymbol{AB} + \boldsymbol{S} + \boldsymbol{W}$.

  ... set $\boldsymbol{A} \sim \mathcal{N}(\boldsymbol{0}, \sigma_A^2 \boldsymbol{I})$, $\boldsymbol{B} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{I})$, and $\boldsymbol{S} \sim$ Bern(λ)-$\mathcal{N}(\boldsymbol{0}, \sigma_S^2 \boldsymbol{I})$.

- **Dictionary Learning** (DL):

  Recover overcomplete $\boldsymbol{A}$ and sparse $\boldsymbol{S}$ from $\boldsymbol{Y} = \boldsymbol{AS} + \boldsymbol{W}$.

  ... set $\boldsymbol{A} \sim \mathcal{N}(\boldsymbol{0}, \sigma_A^2 \boldsymbol{I})$ and $\boldsymbol{S} \sim$ Bern(λ)-$\mathcal{N}(\boldsymbol{0}, \sigma_S^2 \boldsymbol{I})$.

Moreover:

- **Non-Gaussian** (e.g., quantized) observations can be incorporated via $p_{Y|X}$.
- **Structured sparsity** can be incorporated via "**turbo-AMP**." [Schniter'10]
- Hyperparameters can be learned via **EM**. [Ziniel/Schniter'10],[Vila/Schniter'11]

BiG-AMP in Context:

Advantages:

- A **unified** approach to a wide range of problems, e.g., MC, RPCA, DL, ...
- Competitive with best algorithms for each application.
  - **Very fast** and **scaleable**: no SVDs, easily parallelizable.

    ...will see from runtime curves.
  - **Accurate**: in part due to flexibility of choice of priors.

    ...will see from phase transition and NMSE curves.

Relation to other message-passing algorithms for matrix completion:

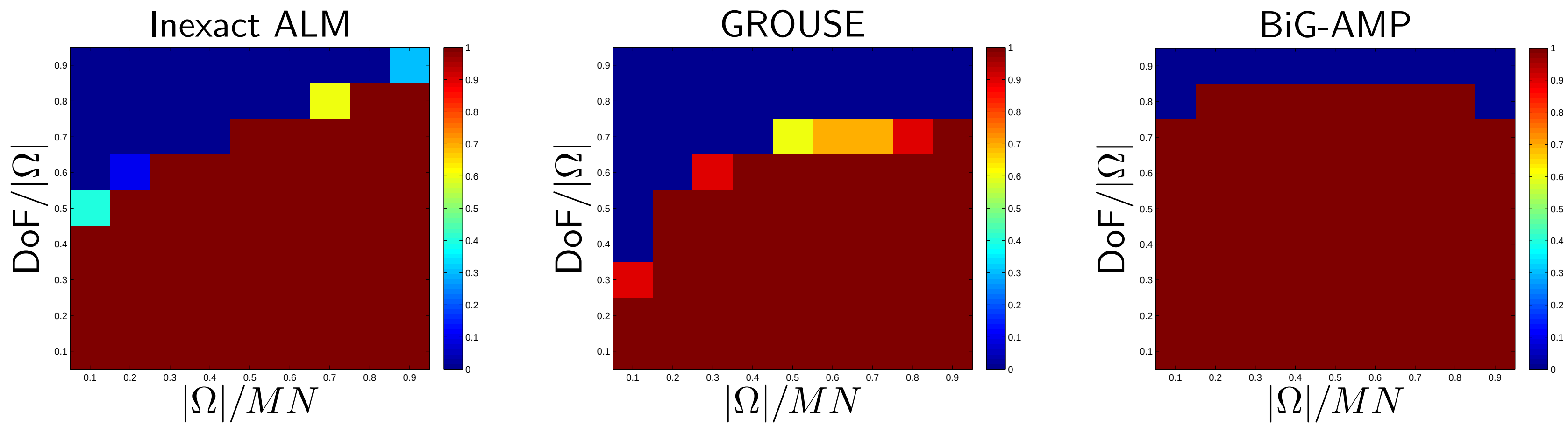
- [Kim/Yedla/Pfister'10]
  - All quantities are **discrete**.
- [Keshavan/Montanari'11] (1 page poster only!)
  - Variable nodes are vector-valued; updates involve **matrix inversion?**

BiG-AMP is a Work-In-Progress:

- Implementation **not yet optimized**.
  - IALM uses ProPack
  - GRASTA uses Mex (5x speedup!)
- Adaptation of **stepsize** not yet implemented.
  - Current stepsize is conservative/slow.
- EM **hyperparameter learning** not implemented.
  - For now, statistics assumed known.
- Many **extensions** to pursue:
  - quantized outputs (e.g., Netflix ratings)
  - non-negativity constraints (e.g., pmf)
  - structure (e.g., tree-structured dictionaries)
  - linear (not missing) observations
  - etc, etc, etc. . .
- **Theoretical analysis/guarantees?**

Matrix Completion — Phase Transitions:

For $M \times N = 512 \times 512$ matrices in the absence of noise, median over 10 trials:



where

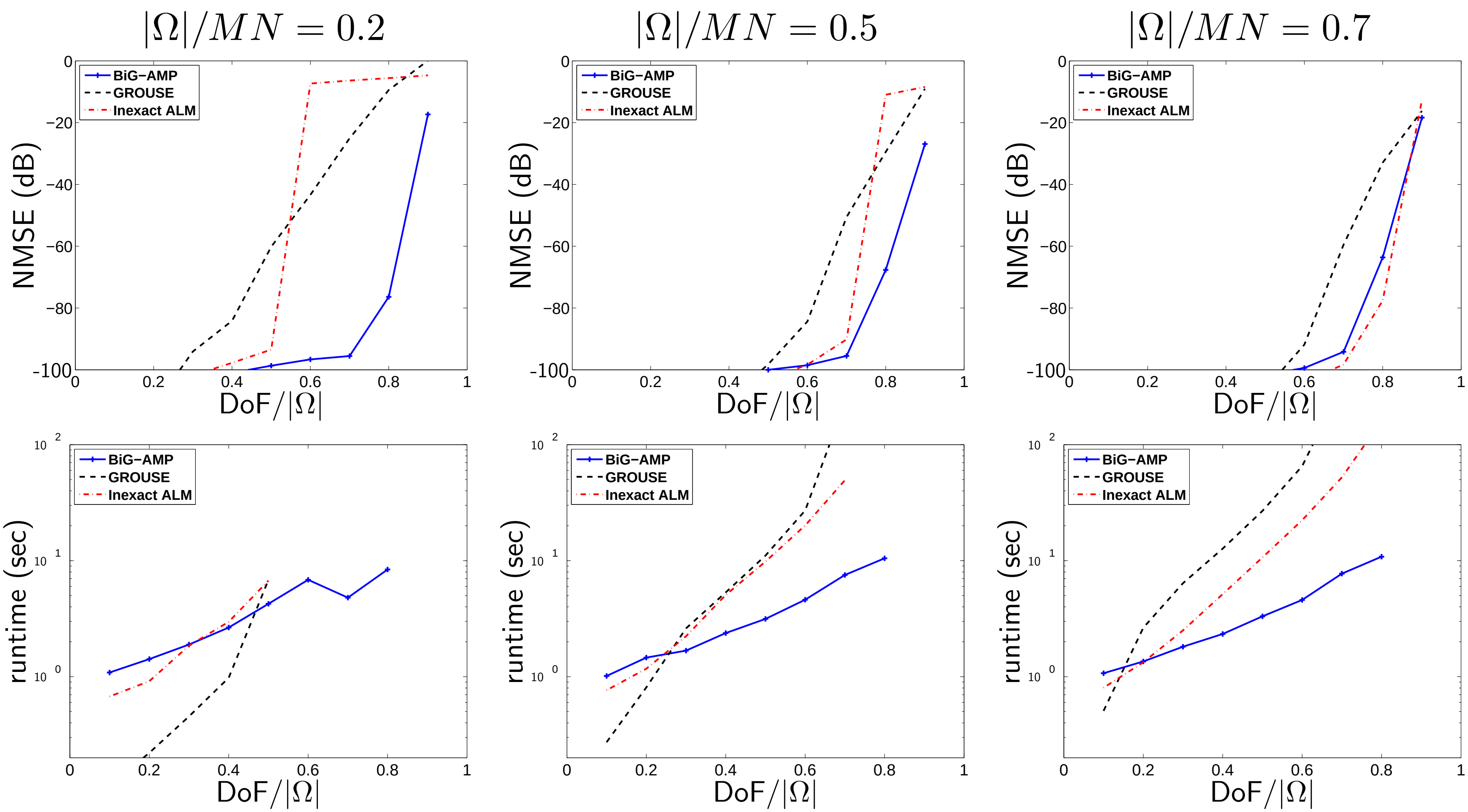
- DoF $\triangleq$ # degrees-of-freedom in SVD (i.e., $MR + RN - R^2$).
- $|\Omega| \triangleq \#$ of observed entries ($|\Omega| \leq MN$).

Observations:

- **BiG-AMP has a better phase transition** than Inexact ALM and GROUSE.
- We are working on understanding its strange behavior near $|\Omega|/MN \approx 1$.

Matrix Completion — NMSE and Runtime (to -50 dB):

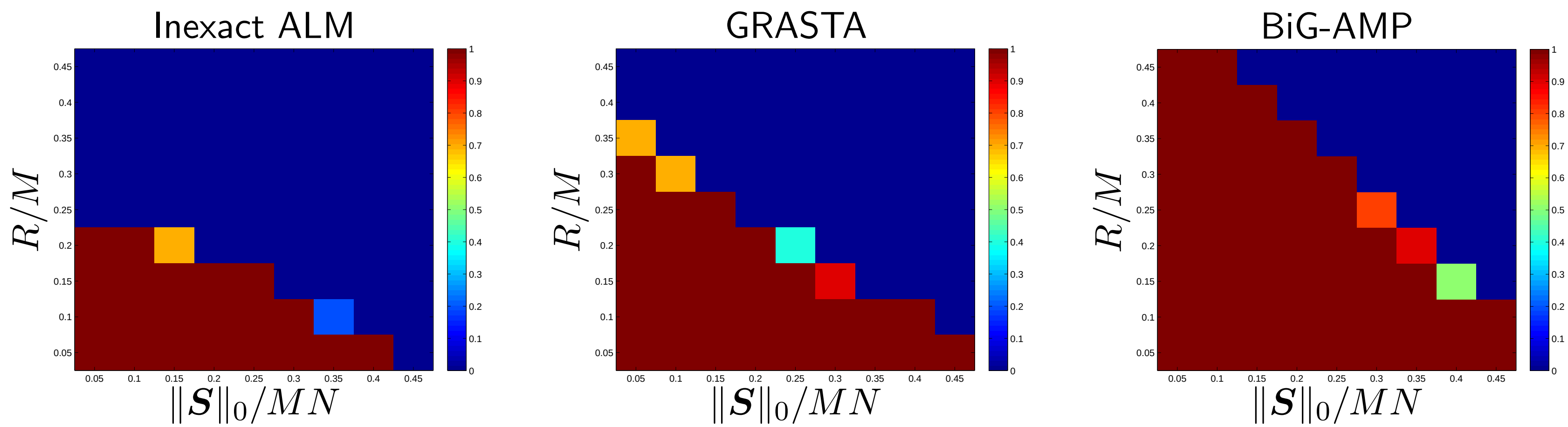
(vertical slices of phase plane)



BiG-AMP is **more accurate** than Inexact ALM and GROUSE, and its **complexity scales better** with rank!

Robust PCA — Phase Transitions:

For $M \times N = 500 \times 500$ matrices in the absence of noise, median over 10 trials:



where

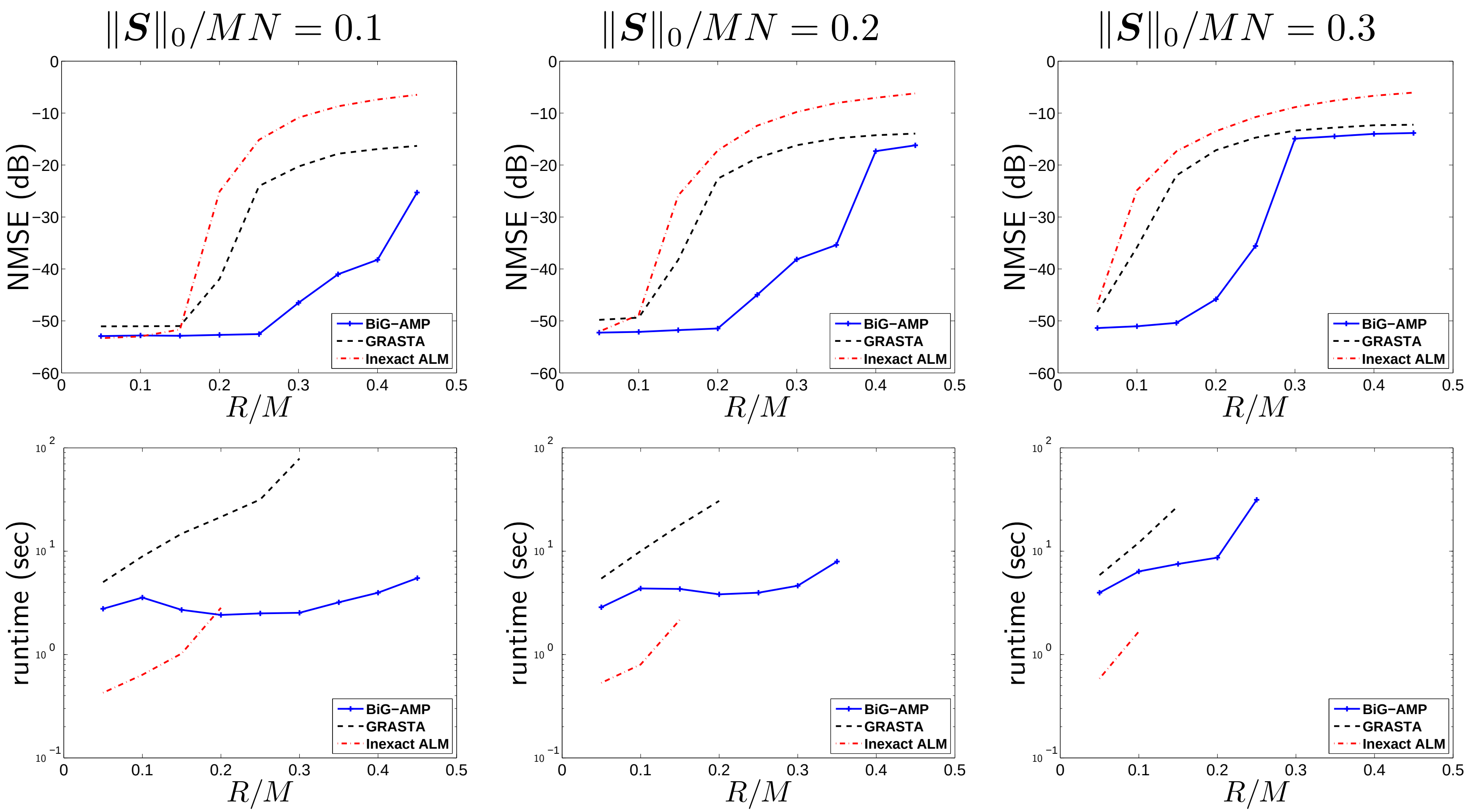
- $R =$ rank.
- $\|\boldsymbol{S}\|_0 = \#$ of corrupted entries.
- $|\Omega| = MN$; all entries observed (although BiG-AMP & GRASTA support $|\Omega| < MN$)

Observations:

- **BiG-AMP has a better phase transition** than Inexact ALM and GRASTA.

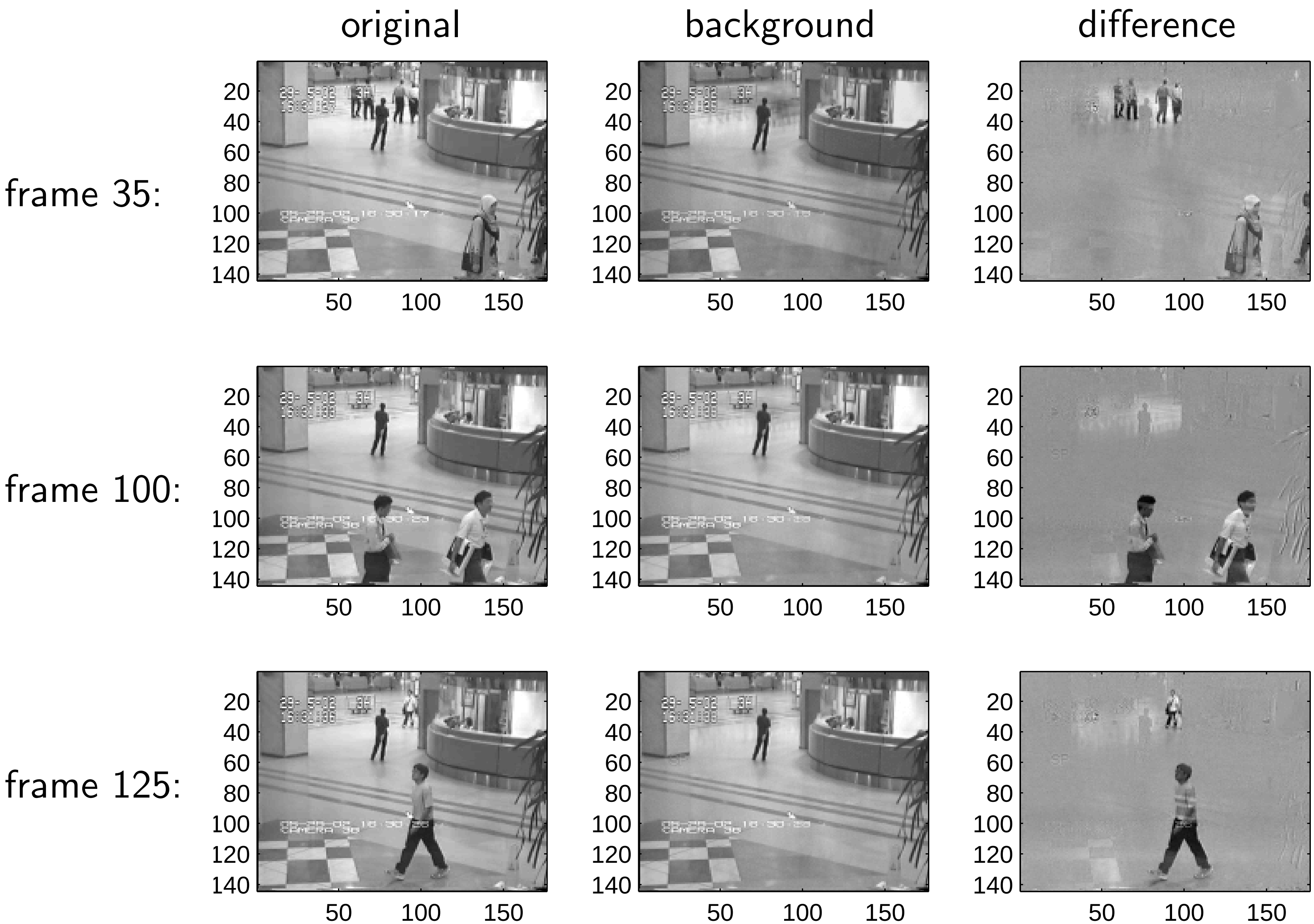
Robust PCA — NMSE and Runtime (to -20 dB):

(vertical slices of phase plane)



BiG-AMP is **more accurate** than Inexact ALM and GRASTA, and its complexity scales better with rank.

Robust PCA — Video Surveillance (over 200 frames):



Conclusions:

BiG-AMP is . . .

- Approximate message passing (**AMP**) for the **generalized bilinear** model.
- A **unified** approach to many **matrix-recovery** problems (MC, RPCA, DL. . .)
- **Competitive** with the best algorithms for each application.
  - Better phase transitions than IALM and GROUSE/GRASTA.
  - Faster on “difficult” problems (e.g., high-rank MC).
- Still a **work in progress** . . . stay tuned!

Special Acknowledgment:

- Many thanks to my PhD student **Jason Parker**!!