Approximate Message Passing for Recovery of Sparse Signals with Markov-Random-Field Support Structure

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The Compressive Sensing Problem

Linear observation of a sparse signal:

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{w}.$$

- $\boldsymbol{A} \in \mathbb{R}^{M \times N}$, a known dense measurement matrix.
- Sparse: $\boldsymbol{x}$ has $K < M$ non-zero coefficients.
- Underdetermined when $M < N$.
- $\boldsymbol{w} \in \mathbb{R}^{M}$ is AWGN $\sim \mathcal{N}(0, \sigma_w^2 \mathbf{I}_M)$.

Our aim is to recover the signal $\boldsymbol{x}$ from observation $\boldsymbol{y}$.

Markov Random Field Structure for Sparsity Pattern

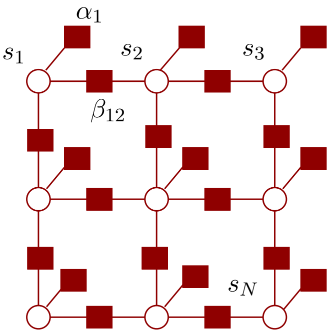
- Indicator vector $\boldsymbol{s} \in \{0,1\}^N$ denotes sparsity pattern.
- Signal coefficients x_n are distributed independently given sparsity variable s_n:

$$p(x_n|s_n) = s_n \mathcal{N}(x_n; 0, \sigma_n^2) + (1 - s_n)\delta(x_n).$$

- The indicator variables are modeled as Markov random field (MRF) $p(\boldsymbol{s})$.
  - A simple MRF model is Ising model:

$$p(\boldsymbol{s}) = \frac{1}{Z} \exp\left(\sum_n s_n \left(\frac{1}{2} \sum_{s_m \in S_n^c} \beta_{mn} s_m - \alpha_n \right) \right).$$

Factor Graph of 2D Ising Model



- Factor graph with loops.

Reconstruction w/ Probabilistically Structured Sparsity

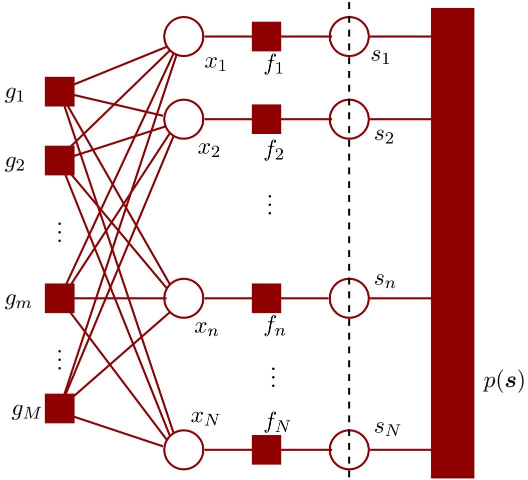
- Markov-chain Monte Carlo (MCMC):
  - Markov random field [Wolfe, Godsill, Ng 2004]
  - Markov tree [He, Carin 2009]

  Drawbacks: slow convergence and difficulty in detecting convergence.
- Methods that iterate matching pursuit or ℓ_1-optimization with MAP sparsity-pattern detection:
  - Markov tree [Duarte, Wakin, Baraniuk 2008]
  - MRF [Cevher, Duarte, Hedge, Baraniuk 2008]

  Drawback: slow and ad hoc.
- Variational Bayes:
  - Markov tree [He, Chen, Carin 2010]

  Drawback: performance not always satisfactory
- Turbo reconstruction based on AMP:
  - Markov chain [Schniter 2010]
  - Markov tree [Som, Potter, Schniter 2010]

Factor Graph Representation



Turbo Reconstruction

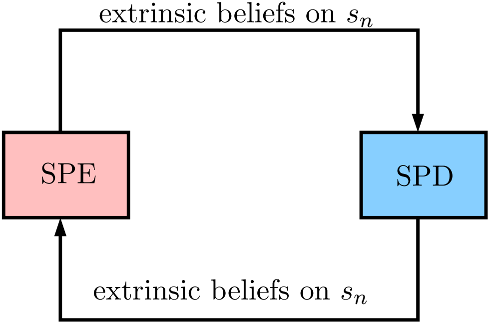
Inference problem can be tackled by *splitting it into two sub-problems and iterating between them*

- Reminiscent of noncoherent turbo equalization.
- The sparsity pattern equalization (SPE) block solves the inference problem using the observation structure (linear observation model).
- The sparsity pattern decoding (SPD) block solves the inference problem using the support structure (Markov model).

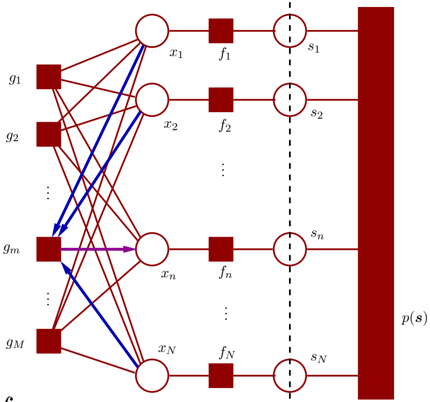
[Schniter 2010]

Message Passing between SPE and SPD

- Message passing within SPE is done via Approximate Message Passing (AMP). [Donoho, Maleki, Montanari 2009]
- SPD is done on MRF using loopy belief propagation algorithm.
- Beliefs on the indicator variables s_n are exchanged between these two blocks.



Gaussian Messages from g_m to x_n

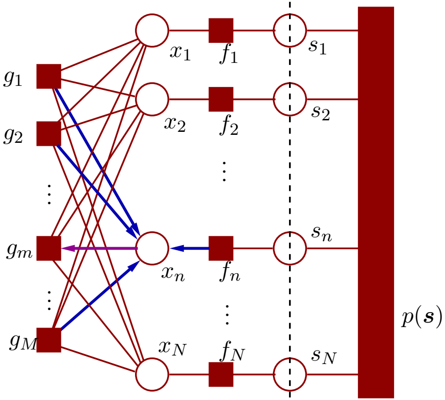


$$\nu_{g_m \to x_n}(x_n) \propto \int_{\{x_q\}_{q \neq n}} \mathcal{N}\Big(y_m; A_{mn}x_n + \sum_{q \neq n} A_{mq}x_q, \sigma_w^2\Big) \prod_{q \neq n} \nu_{x_q \to g_m}(x_q)$$

$$\nu_{g_m \to x_n}(x_n) \quad = \quad \mathcal{N}\left(x_n; \frac{z_{mn}}{A_{mn}}, \frac{c_{mn}}{|A_{mn}|^2}\right)$$

$$z_{mn} = y_m - \sum A_{mq}\mu_{mq} \qquad \text{and} \qquad c_{mn} = \sigma_w^2 + \sum A_{mq}^2 v_{mq}$$

Gaussian Approximated Messages from x_n to g_m



- Outgoing messages are product of incoming messages.
- The (exact) sum-product algorithm would pass non-Gaussian messages, but AMP approximates them as Gaussian
- Computation of means and variances suffice.

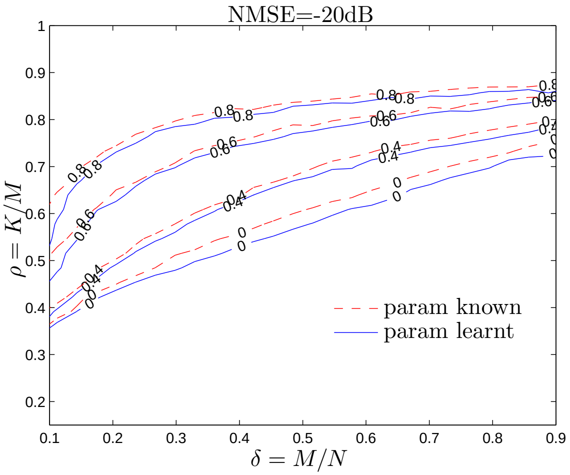
Message Update Complexity

- Message update complexity: MN updates of $\mathcal{O}(N)$ or $\mathcal{O}(M)$ corresponding to MN edges.
- Use two approximations:
  - Apply uniform variance approximations, e.g., $c_n \approx c_{mn}$.
  - Taylor series is used to approximate the deviations of messages across outgoing edges from the average message.
- These approximations reduce the algorithm complexity to $\mathcal{O}(MN)$ per iteration and the number of iterations is small and independent of M, N.
  - For subsampled DFT measurement matrix the complexity is $\mathcal{O}(N \log N)$.

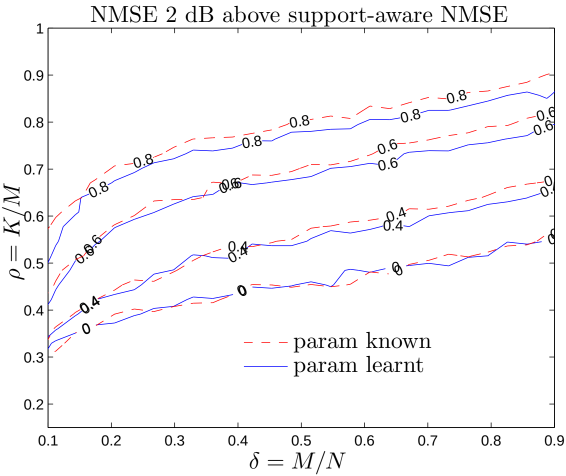
Learning Model Parameters

- After every turbo iterations, model parameters are learnt and updated.
- Signal variance $\sigma^2 \equiv \sigma_n^2$ and noise variance σ_w^2 are learnt using maximum likelihood.
- The Ising model parameters α_n, β_{mn} are learnt by maximizing pseudo-likelihood function. [Besag 1977, 1986]

Empirical Phase Transition Curves



Empirical Phase Transition Curves



Conclusions

- Goal: Recover a sparse signal with a Markov-field structure on the support
- Proposed method: Merge the AMP algorithm with loopy MRF belief propagation using the turbo messaging schedule.
- We propose to learn the model parameters from the measured data.
- We see from the numerical results that signal recovery performance is near the support-oracle bound even when the MRF parameters are apriori unknown.