THE DITHERED SIGNED-ERROR CONSTANT MODULUS ALGORITHM

P. Schniter and C.R. Johnson, Jr.

School of Electrical Engineering
Cornell University
Ithaca, NY 14853 USA
schniter(johnson)@ee.cornell.edu

ABSTRACT

Adaptive blind equalization has gained widespread use in communication systems that operate without training signals. In particular, the Constant Modulus Algorithm (CMA) has become a favorite of practitioners due to its LMS-like complexity and desirable robustness properties. The desire for further reduction in computational complexity has motivated signed-error versions of CMA, which have been found to lack the robustness properties of CMA. This paper presents a simple modification of signed error CMA, based on the judicious use of dither, that results in an algorithm with robustness properties closely resembling those of CMA. An approximation to the steady-state mean-squared error performance of the new algorithm is derived for comparison to that of CMA.

1. INTRODUCTION

The Constant Modulus Algorithm (CMA) [1, 2] has gained widespread practical use as a blind adaptive equalization algorithm for digital communications systems operating over inter-symbol interference channels. Under perfect blind equalizability conditions **(A1)-(A5)** listed in §2.2, CMA has been shown to converge in mean to an equalizer setting capable of perfect symbol recovery.

Though assumptions of ideality are convenient for the theoretical analysis of blind equalization schemes, they are unconditionally violated in physical implementations of communication systems. This fact suggests that the *robustness* of a given blind equalization strategy to violations in perfect conditions may significantly impact its practical worth. In this spirit, it has been reasoned that the widespread practical use of CMA bears testament to its superior robustness properties. A sizeable body of theoretical analysis exists to support this claim (see [3] and references therein).

Though noted for its LMS-like complexity, CMA may be further simplified by transforming the bulk of its update multiplications into sign operations [2]. A recent study suggests that straightforward implementations of signed-error CMA (SE-CMA) do not inherit the desirable robustness properties of CMA [4]. In §3 we present a simple modification of SE-CMA, based on the judicious use of dither, that results in an algorithm with robustness properties closely resembling the standard (unsigned) CMA. The anticipated consequence of dithering is a degradation in steady-state mean-square error (MSE) performance. In §4 we derive an approximation to the excess MSE of dithered SE-CMA which allows

comparison to a similar expression derived for CMA. Implications on convergence rate comparisons are then discussed in §5.

2. FRACTIONALLY-SPACED CMA

2.1. The Fractionally-Spaced System Model

In this paper we consider a noiseless communication system operating at baud interval T. The baseband channel is approximated as linear and FIR, and its $T/2$-spaced impulse response coefficients are collected into the length-N_h vector $\mathbf{h}$. The baseband receiver model is reduced to a $T/2$-spaced linear equalizer described by the N_f coefficients in $\mathbf{f}$. Figure 1 shows the block diagram relating transmitted symbols s_n (indexed by n) to the baud-spaced system outputs y_n.

Defining the fractionally-spaced (FS) convolution matrix $\mathbf{H} =$

$$
\begin{pmatrix}
h_1 & h_0 & & & & \\
h_3 & h_2 & h_1 & h_0 & & & \\
\vdots & \vdots & h_3 & h_2 & \ddots & h_1 & h_0 \\
h_{N_h-1} & h_{N_h-2} & \vdots & \vdots & \ddots & h_3 & h_2 \\
& & h_{N_h-1} & h_{N_h-2} & \ddots & \vdots & \vdots \\
& & & & & h_{N_h-1} & h_{N_h-2}
\end{pmatrix}
$$

allows us to describe the baud-spaced system (mapping $s_n \to y_n$) by its length-N_q impulse response vector $\mathbf{q} = \mathbf{Hf}$. I.e., $y_n = \mathbf{q}^t \mathbf{s}(n)$ for $\mathbf{s}(n) = \left(s_n, s_{n-1}, \ldots, s_{n-N_q+1}\right)^t$. The structure of $\mathbf{H}$ implies that $N_q = \lfloor(N_h + N_f - 1)/2\rfloor$. We define *perfect symbol recovery* (PSR) to mean $y_n = s_{n-\delta}$ for some fixed system delay $0 \le \delta \le N_q - 1$. In such a case, $\mathbf{q} = \mathbf{e}_\delta$ where $\mathbf{e}_\delta$ denotes a vector with 1 in the δth position and zeros elsewhere.

Figure 1: $T/2$-spaced baseband communication system model.

2.2. The Constant Modulus Algorithm

The constant modulus (CM) criterion can be expressed by the cost function $J_{\text{cm}} = \frac{1}{4}E\left\{\left(|y_n|^2 - \gamma\right)^2\right\}$ where γ is a positive constant known as the *Godard radius* [1]. The equalizer update algorithm leading to a stochastic gradient descent of J_{cm} is known as the Constant Modulus Algorithm (CMA) and is specified by [2]

$$
\mathbf{f}(n+1) = \mathbf{f}(n) + \mu \mathbf{r}^*(n) \underbrace{y_n\left(\gamma - |y_n|^2\right)}_{\triangleq \psi(y_n)} \tag{1}
$$

where μ is a step-size and $\mathbf{r}(n)$ is the equalizer input vector at time index n. The asterisk denotes conjugation. The function $\psi(\cdot)$ identified in (1) is referred to as the *CMA error function*.

The following *perfect blind equalizability* (PBE) conditions are known (e.g. [3]) to be sufficient to guarantee that equalizers minimizing J_{cm} achieve perfect symbol recovery.

(A1) Sufficient equalizer length: For a $T/2$-spaced FSE, $N_f \geq 2\lceil N_h/2 \rceil - 2$.

(A2) Subchannel disparity: The polynomials formed from the even and odd coefficients of $\mathbf{h}$ must share no common roots.

(A3) No additive channel noise.

(A4) Sub-Gaussian source: The source kurtosis $\kappa_s = \frac{E\{|s_n|^4\}}{E\{|s_n|^2\}^2}$ must be less than that of a Gaussian process.

(A5) White source: The source symbols must be temporally uncorrelated (and when complex-valued, $E\{s_n^2\} = 0$).

Note that **(A4)-(A5)** pertain to blind equalization via the CM criterion, while **(A1)-(A3)** are required[1] to guarantee perfect symbol recovery for a given channel-equalizer combination.

3. COMPUTATIONALLY EFFICIENT CMA

Straightforward implementations of LMS-like adaptive algorithms (such as CMA) require a multiply the between the error function and every regressor element (see (1)). Many practical applications benefit from eliminating these N_f regressor multiplies. *Signed-error* (SE) algorithms present one method for doing so, whereby only the sign of the error function is retained [5]. When a SE algorithm is combined with a power-of-two step-size, it is possible to construct multiply-free fixed-point implementations of the equalizer update algorithm. The sections below discuss two versions of SE-CMA. For the remainder of the paper, we restrict our focus to the case where all quantities are real-valued.

3.1. Signed-Error CMA

The real-valued SE-CMA algorithm is specified by [2]

$$\mathbf{f}(n+1) \quad = \quad \mathbf{f}(n) + \mu \mathbf{r}(n) \underbrace{\operatorname{sgn}\left(y_n(\gamma - y_n^2)\right)}_{\triangleq \sigma(y_n)} \qquad (2)$$

where $\operatorname{sgn}(\cdot)$ is the standard (real-valued) signum function. Figure 2 compares signed and unsigned versions of the CMA error function: σ and ψ, respectively.

A recent investigation into SE-CMA has shown that, while satisfaction of the PBE conditions and correct selection of γ ensures convergence to a PSR setting, violation of **(A1)** severely hinders SE-CMA convergence behavior [4]. Specifically, there exist vast yet highly suboptimal regions in equalizer space for which the average update in (2) is zero. Thus, while computationally efficient, SE-CMA does not inherit the desirable robustness properties of CMA. This motivates the search for computationally efficient blind algorithms which *do* inherit these robustness properties. The following section describes one such algorithm.

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[1] We acknowledge the existence of peculiar violations of **(A1)-(A2)** allowing the possibility of perfect symbol recovery for a restricted range of δ [3], but dismiss them on account of their academic nature.
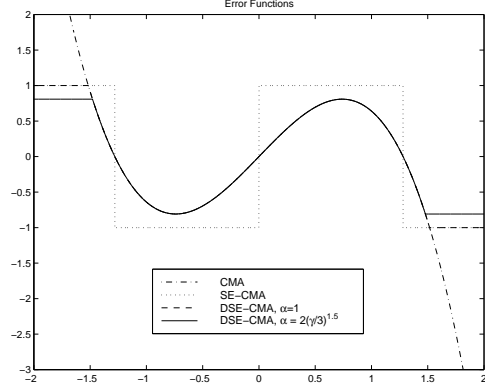
Figure 2: CMA, SE-CMA, and DSE-CMA error functions.

3.2. Dithered Signed-Error CMA

In this section we describe a simple modification to SE-CMA that results in an algorithm whose average behavior closely matches that of (unsigned) CMA.

The one-bit quantization inherent to signed-error adaptation algorithms motivates the application of dither [7]. Dithering techniques attempt to preserve information lost in the quantization process by making the quantization noise white, zero-mean, and independent of the signal being quantized. Intuitively, a small step-size adaptive algorithm can then "average out" the quantization noise, yielding mean behavior nearly identical to the unsigned algorithm. See [6] for an example of adding controlled noise to SE-LMS for echo cancellation.

We define real-valued *dithered SE-CMA* (DSE-CMA) as:

$$\mathbf{f}(n+1) \quad = \quad \mathbf{f}(n) + \mu \mathbf{r}(n) \underbrace{\alpha \operatorname{sgn}\left(\psi(n) + \alpha d_n\right)}_{\triangleq \varphi_\alpha(y_n, d_n)} \qquad (3)$$

where d_n are samples of the dither process, α is a real positive constant, and $\psi(n)$ is the CMA error function defined in (1).

For this application, $\{d_n\}$ is required to be an i.i.d. random process whose characteristic function has zeros at all multiples of π except the origin [7]. In other words $E\{e^{j\pi l d_n}\} = 0 \quad \forall l \neq 0$. With these properties, the theorems in [7] imply that the expected DSE-CMA error function is a "hard-limited" version of the CMA error function:

$$\varphi_\alpha(y_n) \triangleq E\{\varphi_\alpha(y_n, d_n)|y_n\} = \begin{cases} \alpha & y_n : \psi(y_n) > \alpha, \\ \psi(y_n) & y_n : |\psi(y_n)| \leq \alpha, \\ -\alpha & y_n : \psi(y_n) < -\alpha. \end{cases}$$

See Figure 2 for a plot of $\varphi_\alpha(\cdot)$ for various values of α. Note that $\alpha \geq 2(\gamma/3)^{3/2}$ prevents the two "humps" of the CMA error function from being clipped. Thus, when $\alpha = 2(\gamma/3)^{3/2}$, we expect the average update behavior of DSE-CMA to be identical to that of CMA for all equalizers satisfying the output constraint $|y_n| \leq 2\sqrt{\gamma/3}$, since $|\psi(y_n)| \leq \alpha$ for these y_n.

The relationship between the DSE-CMA and CMA error functions for any particular $\alpha \geq 2(\gamma/3)^{3/2}$ implies that the respective cost functions have identical shape within the convex polytope of equalizers satisfying a particular output amplitude constraint (see Fig. 3). This constraint was stated earlier for $\alpha = 2(\gamma/3)^{3/2}$; for
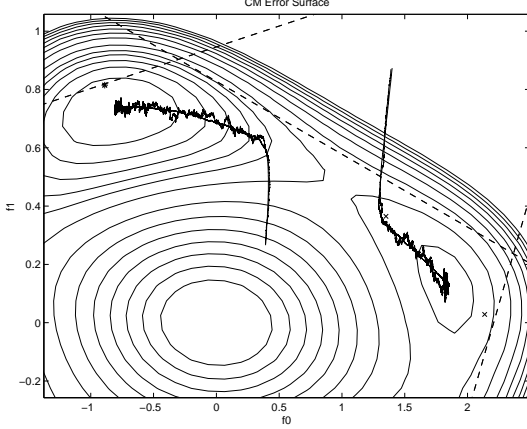
Figure 3: Trajectories of DSE-CMA (rough) overlaid on those of CMA (smooth) for BPSK, noiseless channel $\mathbf{h} = (0.1, 0.3, 1, -0.1, 0.5, 0.2)^t$, $\mu = 5 \times 10^{-4}$, $\alpha = 1$, and uniform $\{d_n\}$. Solid lines are J_{cm} contours, and dashed lines bound the convex output-amplitude-constraint polytope within which the CMA and DSE-CMA cost functions are identically shaped.

arbitrary $\alpha > 2(\gamma/3)^{3/2}$, the bound on $|y_n|$ is given by the the largest root of the polynomial $y(\gamma - y^2) - \alpha$.

Figure 3 shows two examples of DSE-CMA trajectories overlaid on CMA trajectories initialized at the same locations. Note that the DSE-CMA trajectories exhibit much more parameter variation than the CMA trajectories. The effect of this parameter variation on steady-state performance is quantified in §4.3.

It is worth mentioning that, of all dither processes, a uniform distribution on $(-1, 1]$ leads to the lowest quantization noise power [7] and hence the lowest parameter variance.

3.3. Selection of the Godard Radius: The General Case

This section outlines a procedure which can be used to choose the Godard radius γ given an arbitrary error function such as φ_α. We follow the method of Godard in [1], whereby γ is selected to force the mean equalizer update to zero when perfect equalization has been achieved. Taking DSE-CMA as our example, the mean update term is $\mu \mathbf{r}(n)\varphi_\alpha(y_n)$ from (3). From §2.1, we know that $\mathbf{r}(n) = \mathbf{H}^t\mathbf{s}(n)$ and that, at perfect symbol recovery, $y_n = s_{n-\delta}$. Assuming an i.i.d. source process, $\varphi_\alpha(s_{n-\delta})$ is independent of all but one element in $\mathbf{s}(n)$, namely $s_{n-\delta}$. Hence, for a zero update, we only require that the value of γ in φ_α be chosen so that

$$E\{\varphi_\alpha(s_n)s_n\} = 0 \qquad (4)$$

For the CMA algorithm of (1), it is well know that this procedure yields $\gamma = E\{s^4\}/E\{s^2\}$. In [4], the authors give an expression for γ in the case of M-PAM (real-valued) SE-CMA. Closed form expressions for γ in the case of M-PAM DSE-CMA with arbitrary α are difficult if not impossible to derive. Fortunately, for finite-alphabet sources, γ satisfying (4) can easily be determined numerically.

4. STEADY-STATE BEHAVIOR OF DSE-CMA

As noticed earlier, the principle disadvantage of DSE-CMA concerns its steady-state behavior: the addition of dither leads to an increase in *excess mean-squared error* (EMSE). We define EMSE as the steady-state MSE above that achieved by the (locally) optimal fixed parameter setting. In the ensuing analysis we assume that the PBE conditions are satisfied, in which case the minimum achievable MSE is zero.

The subsections below attempt to quantify the excess MSE of DSE-CMA when the PBE conditions are satisfied. The approach taken is the following: in the vicinity of a minima, the CMA cost function is well approximated by a quadratic error surface, implying that the steady-state behavior of CMA can be linked to the steady-state behavior of LMS. Existing results on the asymptotic parameter distribution of fixed step-size SE-LMS are then applied to describe the steady-state parameter distribution of DSE-CMA, from which an approximation of the EMSE is derived. Specifically, these results imply that the DSE-CMA parameters asymptotically approach i.i.d. Gaussian random variables [8].

4.1. Local Approximation of CM Cost

Assume, w.l.o.g., a unit variance source: $E\{s^2\} = 1$. Then the CM cost in terms of the system parameters $\mathbf{q}$ is [3]

$$J_{\text{cm}}(\mathbf{q}) = \frac{\kappa_s - 3}{4} \sum_{i=0}^{N_q - 1} q_i^4 + \frac{3}{4}\|\mathbf{q}\|_2^4 - \frac{\kappa_s}{2}\|\mathbf{q}\|_2^2 + \frac{\kappa_s^2}{4} \qquad (5)$$

where κ_s was defined in (A4). The second-order Taylor series expansion of J_{cm} about the minimum $\mathbf{q} = \mathbf{e}_\delta$ can be obtained by straightforward vector calculus. (Details will be provided by the author upon request.) Defining the optimal equalizer $\mathbf{f}_\delta = \mathbf{H}^{-1}\mathbf{e}_\delta$ and the parameter error $\tilde{\mathbf{f}} = \mathbf{f} - \mathbf{f}_\delta$, Taylor's theorem suggests that J_{cm} is well approximated by $\tilde{J}_{\text{cm}}(\tilde{\mathbf{f}}) =$

$$\frac{\kappa_s(\kappa_s - 1)}{4} + \frac{3 - \kappa_s}{2}\tilde{\mathbf{f}}^t\mathbf{H}^t\mathbf{H}\tilde{\mathbf{f}} + \frac{3(\kappa_s - 1)}{2}\tilde{\mathbf{f}}^t\mathbf{H}^t\mathbf{e}_\delta\mathbf{e}_\delta^t\mathbf{H}\tilde{\mathbf{f}} \quad (6)$$

for small $\tilde{\mathbf{f}}$.

The last term in (6) is a quadratic form involving the non-Toeplitz matrix $\mathbf{H}^t\mathbf{e}_\delta\mathbf{e}_\delta^t\mathbf{H}$. As we desire $\tilde{J}_{\text{cm}}$ of the form $\tilde{J}_{\text{cm}}(\tilde{\mathbf{f}}) = J_{\min} + \tilde{\mathbf{f}}^t\mathbf{R}_{\mathbf{x},\mathbf{x}}\tilde{\mathbf{f}}$, where $J_{\min}$ is a constant and $\mathbf{R}_{\mathbf{x},\mathbf{x}}$ is the autocorrelation matrix of some stationary ergodic equalizer input process $\{\mathbf{x}\}$, we will further approximate (6).

Using the following facts: $\text{tr}(A) = A$ for any scalar A, $\text{tr}(\tilde{\mathbf{f}}^t\mathbf{A}\tilde{\mathbf{f}}) = \text{tr}(\tilde{\mathbf{f}}\tilde{\mathbf{f}}^t\mathbf{A})$ and $E\{\text{tr}(\mathbf{A})\} = \text{tr}(E\{\mathbf{A}\})$ for any matrix $\mathbf{A}$, and $E\{\tilde{\mathbf{f}}\tilde{\mathbf{f}}^t\} = C\mathbf{I}_{N_f}$ for some scalar C [8], we claim

$$\begin{aligned} E\{\tilde{\mathbf{f}}^t\mathbf{H}^t\mathbf{e}_\delta\mathbf{e}_\delta^t\mathbf{H}\tilde{\mathbf{f}}\} &= C\,\text{tr}(\mathbf{e}_\delta^t\mathbf{H}\mathbf{H}^t\mathbf{e}_\delta) \\ &= C\,\|(h_{2\delta - N_f + 2}, \ldots, h_{2\delta+1})\|_2^2 \\ &\approx \frac{C}{2}\|\mathbf{h}\|_2^2 \end{aligned}$$

where the approximation is based on the structure of $\mathbf{H}$ from §2.1. Since, by the same techniques, $E\{\tilde{\mathbf{f}}^t\mathbf{H}^t\mathbf{H}\tilde{\mathbf{f}}\} = C\,\text{tr}(\mathbf{H}\mathbf{H}^t) = \frac{CN_f}{2}\|\mathbf{h}\|_2^2$, we can approximate $\tilde{J}_{\text{cm}}$ by

$$\hat{J}_{\text{cm}} = \frac{\kappa_s(\kappa_s - 1)}{4} + \frac{1}{2}\big(3 - \kappa_s + 3(\kappa_s - 1)/N_f\big)\tilde{\mathbf{f}}^t\mathbf{H}^t\mathbf{H}\tilde{\mathbf{f}} \quad (7)$$

4.2. Asymptotic Parameter Distribution

Linking the locally approximated CM cost to an LMS update will allow us to use existing results on the steady-state parameter distribution of SE-LMS to approximate that of DSE-CMA.

Applying LMS to the equalization problem of Fig. 1 yields

$$\tilde{\mathbf{f}}(n+1) \quad = \quad \tilde{\mathbf{f}}(n) + \mu \mathbf{r}(n)\big(e_n + u_n\big) \qquad (8)$$

where $e_n = s_{n-\delta} - y_n$ is an error signal driven to zero when $\tilde{\mathbf{f}} = 0$, and u_n is a non-vanishing "noise" process [5]. The LMS algorithm is known to stochastically minimize

$$J_{\text{lms}} \;=\; J_{\min} + \frac{1}{2}\tilde{\mathbf{f}}^t \mathbf{R}_{\mathbf{r},\mathbf{r}} \tilde{\mathbf{f}} \;=\; J_{\min} + \frac{1}{2}\tilde{\mathbf{f}}^t \mathbf{H}^t \mathbf{H} \tilde{\mathbf{f}}$$

Comparing J_{lms} to $\hat{J}_{\text{cm}}$, we conclude that CMA is well approximated by the LMS-like recurrence

$$\tilde{\mathbf{f}}(n+1) \quad = \quad \tilde{\mathbf{f}}(n) + \mu \mathbf{r}(n)\big(K e_n + u_n\big) \qquad (9)$$

for small $\tilde{\mathbf{f}}$, where the gain

$$K \quad = \quad 3 - \kappa_s + 3(\kappa_s - 1)/N_f \qquad (10)$$

accounts for a slope calibration between J_{lms} and $\hat{J}_{\text{cm}}$, and where u_n accounts for the minimum achievable CM cost $J_{\min} = \kappa_s(\kappa_s - 1)/4$. Comparing (1) with $y_n = s_{n-\delta}$ to (9) with $\tilde{\mathbf{f}} = 0$, it is evident that $\{u_n\}$ must take on the values $\{\psi(s_n)\}$ normalized so that $\sigma_u^2 = \kappa_s(\kappa_s - 1)/4$. In short, u_n represents the noisy effect that a non-CM source has on the CMA update (see [9, 3]).

Extending (9) to its dithered signed-error version (and dividing the argument of the $\text{sgn}(\cdot)$ operation by the positive constant K) gives a local approximation to DSE-CMA for small $\tilde{\mathbf{f}}$:

$$\tilde{\mathbf{f}}(n+1) \;=\; \tilde{\mathbf{f}}(n) + \mu\,\alpha\mathbf{r}(n)\,\text{sgn}\big(e_n + (u_n + \alpha d_n)/K\big) \quad (11)$$

Adaptation algorithms of the form (11) have been shown (under certain conditions) to have parameter distributions which are asymptotically Gaussian with covariance matrix [8]

$$\frac{\mu\alpha}{4K p_{u+\alpha d}(0)}\mathbf{I}_{N_f} \qquad (12)$$

where $p_{u+\alpha d}(0)$ is the probability density function of the random process $\{u_n + \alpha d_n\}$ evaluated at the point 0.

One important condition on (12) is that $p_{u+\alpha d}$ must be smooth and bounded with $p_{u+\alpha d}(0) > 0$ [8]. This condition imposes a $\{u_n\}$-dependent lower bound on α for which (12) remains valid and prevents this analysis from directly applying to (non-dithered) SE-CMA, since $p_{\psi(s)}(0) = 0$ for a finite-alphabet source.

Another condition on (12) is that $\{u_n\}$ must be statistically independent of $\mathbf{r}(n)$. This is certainly not true for $u_n = \psi(s_{n-\delta})$ since $\mathbf{r}(n) = \mathbf{H}^t\mathbf{s}(n)$. Thus, we restrict our calculation of EMSE in §4.3 to a constant modulus source (i.e. BPSK), so that $u_n = 0$.

4.3. Excess MSE (Under PBE Conditions)

Given the steady-state parameter covariance matrix (12), it is possible to calculate the mean-squared error $J_{\text{mse}} = E\big\{(y_n - s_{n-\delta})^2\big\}$ $= E\big\{\tilde{\mathbf{f}}^t\mathbf{H}^t\mathbf{H}\tilde{\mathbf{f}}\big\}$. It was shown in §4.1 that, when $E\big\{\tilde{\mathbf{f}}\tilde{\mathbf{f}}^t\big\} = C\mathbf{I}$, $E\big\{\tilde{\mathbf{f}}^t\mathbf{H}^t\mathbf{H}\tilde{\mathbf{f}}\big\} = \frac{C N_f}{2}\|\mathbf{h}\|_2^2$. Restricting focus to BPSK, equations (10) and (12) with $\kappa_s = 1$ specify C, thus giving

$$J_{\text{mse}} \;\approx\; \mu\alpha N_f\|\mathbf{h}\|_2^2/16 p_{\alpha d}(0) \qquad (13)$$

When $\{d_n\}$ is uniformly distributed on $(-1, 1]$, we know that $p_{\alpha d}(0) = p_d(0)/\alpha = 2/\alpha$, and so (13) becomes

$$J_{\text{mse}} \;\approx\; \mu\alpha^2 N_f\|\mathbf{h}\|_2^2/8 \qquad (14)$$

Table 1: Approximate scale factor to apply to (14) for M-PAM.

M	2	4	8	16	32
factor	1	1.3	1.7	3.4	3.5

As discussed in §3.2, higher values of α enlarge the convex polytope within which the shape of CMA and DSE-CMA cost functions is identical (see Fig. 3). As evident from (14), however, α has a squared effect on EMSE. Hence, the selection of α is a design tradeoff between CMA-like robustness and EMSE.

Given that there exists a formula for the EMSE of CMA [9]:

$$J_{\text{mse}} \quad = \quad \frac{\mu N_f\|\mathbf{h}\|_2^2}{4(3 - \kappa_s)}\left(\frac{E\{s_n^6\}}{E\{s_n^2\}^3} - \kappa_s^2\right)E\{s_n^2\}^2 \quad (15)$$

it is unfortunate that our preliminary results on DSE-CMA only pertain to BPSK. (A general expression for M-PAM case is being derived.) SPIB-based (http://spib.rice.edu/) microwave channel simulations seem to indicate, however, that non-CM sources scale the EMSE of (14) proportionally. Table 1 estimates this scale factor for various M-PAM constellations when $\alpha = 1$.

5. CONCLUSIONS

With hardware cost in mind, CMA implementations often update the equalizer coefficients only once per N_f equalizer input samples, allowing one multiplier to time-share the N_f regressor multiplies. Assuming this scheme decreases convergence rate by a factor of N_f, the results of §4.3 indicate that DSE-CMA constitutes a worthwhile improvement over "one-multiplier CMA" for reasonable equalizer lengths N_f. A more detailed study will follow.

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