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ON MMSE FRACTIONALLY-SPACED EQUALIZER DESIGN*

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Abstract: The objective of this paper is to add analytical tools to an evolving theory on fractionally-spaced equalizer (FSE) design[1]. The work examines minimum mean square error design (MMSE-FSE) for channels perturbed by additive noise in two scenarios: when the equalizer is finite, but is at least as long as the channel, and when the equalizer has an infinite impulse response (IIR). Mean square error (MSE) expressions in terms of channel parameters, roots, and spectra are derived as analytical tools for understanding the intricacies of fractionally-spaced equalization and for developing guidelines towards design.

1. MOTIVATION

Recent results [10] have shown that under the **zero & length** conditions (see §2.2), a finite impulse response (FIR) FSE can be used for perfect equalization in the absence of noise. In [6], it is shown that the MMSE-FSE is an IIR filter. The IIR solution can be viewed as the classical components of matched (numerator) and whitening (denominator) filters. Practical considerations dictate that an FIR filter be used for the equalizer, especially in an adaptive application. The variations in the FIR solutions between the noiseless condition and that of typical SNRs present in digital communications (e.g., SNR $>$ 20dB) merit the analysis we present. Also, at high SNRs a sufficiently long FIR MMSE-FSE can almost achieve ideal IIR FSE-MMSE performance. Further motivation for our research lies in the recent wealth of analysis pertaining to blind adaptive equalizers [8] and FIR length choice [7, 11] for FSE equalizers. Although the most common justification for FSEs has been the improved timing recovery property [12], these recent results show improved performance by an FSE with respect to a solution space and adaptive behavior. However, the performance prediction versus length, delay and SNR remains unclear in the literature.

Our intent in this paper is to present new findings on MMSE-FSE ($T/2$-spaced)[1] designs that can be used to address this admissibility question. In particular, we (i) derive an approximation to the achieved MSE of an (FIR) equalizer matching the channel length for a particular SNR in terms of the channel roots, (ii) find closed form expressions for the **perfect length** FSE zero forcing solution and for (iii) the infinite impulse response Wiener receiver (IIR MMSE-FSE) [6] and (iv) using this expression for the solution, compute bounds on the achieved MSE of finite length FSEs longer than the channel time span.

2. FIR MMSE-FSE DESIGN

2.1. Communication system model

We are interested in the multichannel communication system model for fractionally-spaced equalization [9] shown in Figure 1. A T-spaced symbol sequence $\{s(k)\}_{k\in\mathbf{Z}}$,

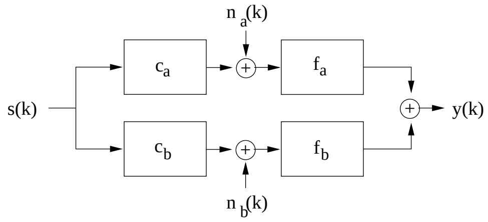

Figure 1. FSE communications model.

$s(k) \in \mathbf{R}$ i.i.d., is transmitted over a FIR, real-valued, T-spaced subchannels (here z^{-1} corresponds to a T-spaced delay)

$$c_a(z) = \sum_{i=0}^{n} c_a^{(i)} z^{-i}, \;\; c_b(z) = \sum_{i=0}^{n} c_b^{(i)} z^{-i}.$$

The channel model includes additive white Gaussian noise (AWGN) $\{n_a(k)\}_{k\in\mathbf{Z}}$, $\{n_b(k)\}_{k\in\mathbf{Z}}$, with $n_a, n_b \in \mathbf{R}$ uncorrelated with each other and uncorrelated with the symbols $s(k)$. Each T-spaced subchannel $c_a(z)$ and $c_b(z)$ is connected to T-spaced subfilters

$$f_a(z) = \sum_{i=0}^{m} f_a^{(i)} z^{-i}, \;\; f_b(z) = \sum_{i=0}^{m} f_b^{(i)} z^{-i}.$$

Let us define a symbol vector $S(k) = (s(k), s(k-1), \ldots, s(k-p))^t$, with $p = n+m$, a noise vector $N(k) = (n_a(k), n_a(k-1), \ldots, n_a(k-m), \;\; n_b(k), n_b(k-1), \ldots, n_b(k-m))^t$, and the equalizer column vector of coefficients $f = \begin{pmatrix} f_a \\ f_b \end{pmatrix}$.

We also define the fractionally-spaced convolution matrix in Sylvester form

$$\mathcal{C} = \left(\begin{array}{cccccccc}
c_a^{(0)} & & & & c_b^{(0)} & & & \\
c_a^{(1)} & c_a^{(0)} & & & c_b^{(1)} & c_b^{(0)} & & \\
\vdots & c_a^{(1)} & & & \vdots & c_b^{(1)} & & \\
c_a^{(n)} & \vdots & \ddots & c_a^{(0)} & c_b^{(n)} & \vdots & \ddots & c_b^{(0)} \\
 & c_a^{(n)} & & c_a^{(1)} & & c_b^{(n)} & & c_b^{(1)} \\
 & & & \vdots & & & & \vdots \\
 & & & c_a^{(n)} & & & & c_b^{(n)}
\end{array} \right).$$

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[1]We will not be considering the general T/L design, and hence will be referring to the $T/2$-spaced equalizers as fractionally-spaced equalizers.

The receiver output is given by

$$y(k) \quad = \quad S^t(k)\mathcal{C}f + N^t(k)f.$$

Our assumptions are:

A1 The channel is assumed to be (linear) FIR, time-invariant, real, and have an even number of taps.

A2 The input is real and i.i.d.

A3 The noise is AWGN and uncorrelated with the source. Subchannel noise sequences are uncorrelated. SNR is always greater than unity (or 0 dB).

A4 The fractionally-spaced equalizer is FIR.

2.2. MSE expressions

Recall that our initial goal was to analyze the achieved MSE of MMSE-FSE designs. In this section we study an approximation to the MSE for the FIR, high SNR case. The requirements on SNR are discussed later.

Define the symbol recovery error $e(k) = s(k-\delta) - y(k)$. The MSE is then given by

$$\sigma_e^2(f,\delta) \quad = \quad \mathrm{E}\{e(k)^2\} \tag{1}$$

for some delay $0 \le \delta \le p$. Assume (w.l.o.g.) that the symbol sequence power is $\sigma_s^2 = 1$, let the noise power be σ_n^2, and define the ratio $\gamma = \frac{\sigma_n^2}{\sigma_s^2}$ as the inverse of the SNR. By defining $t_\delta = (0,\ldots,0,1,0\ldots,0)^t$ with the 1 at position δ, $0 \le \delta \le p$, one can show [7] using **A1-A4** that that $\sigma_e^2(f,\delta)$ is minimized by the Weiner, or FIR MMSE-FSE receiver

$$f_\delta^\star \quad = \quad (\mathcal{C}^t\mathcal{C} + \gamma I)^{-1}\mathcal{C}^t t_\delta \tag{2}$$

for some fixed delay δ. In order to minimize (1) over δ, one can substitute (2) into (1), and obtain a different expression for the MSE

$$\sigma_e^2(f_\delta^\star,\delta) \quad = \quad t_\delta^t\left(I - \mathcal{C}(\mathcal{C}^t\mathcal{C}+\gamma I)^{-1}\mathcal{C}^t\right)t_\delta. \tag{3}$$

Note that the MSE for a certain delay δ in (3) is given by the diagonal entries of the matrix in large parentheses. We will attempt to extract from this last expression an approximation which provides insight to the problem.

To do this, we follow the analysis in [5], and make the **zero & length** assumption:

A5 (zero & length) The subchannels have no common roots and the equalizer is sufficiently long. More precisely, (**zero** or **disparity**) $\gcd(c_a(z), c_b(z)) = 1$ and (**length**) $m \ge n-1$.

Using the matrix inversion lemma[2] and a power series expansion[3] in γ as done in [5], we find

$$\sigma_e^2(f_\delta^\star,\delta) \quad = \quad \gamma\, t_\delta^t(\mathcal{C}\mathcal{C}^t)^{-1}t_\delta + o(\gamma).$$

which is valid when γ is small, more precisely when $\gamma \ll \lambda_{\min}(\mathcal{C}\mathcal{C}^t)$.[4] This suggests the approximation $\sigma_e^2 \approx \hat{\sigma}_e^2 = \gamma\left[(\mathcal{C}\mathcal{C}^t)^{-1}\right]_{\delta,\delta}$, which is relatively simple, but does not provide immediate intuition.

[2] $(BCD+A)^{-1} = A^{-1} - A^{-1}B(DA^{-1}B + C^{-1})^{-1}DA^{-1}$

[3] $(I - D^{-1}A)^{-1} = \sum_{k=0}^{\infty}(D^{-1}A)^k$

[4] Notation: $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ are the minimum and maximum eigenvalues of (square) matrix A respectively. $\lambda(A)$ is any eigenvalue of matrix A.

The approximation of the MSE is related to the product of the difference of all possible subchannel pairs. [5] proposes as a measure of channel disparity the function

$$\psi \quad = \quad \gamma\left(\prod_i |f_a(\beta_i)| \prod_j |f_b(\alpha_j)| \prod_{i,j} |\alpha_i - \beta_j|\right)^{-2}$$

where $\{\alpha_i\}$ are the roots of $c_a(z)$ and $\{\beta_i\}$ are the roots of $c_b(z)$. The measure assumes the value $+\infty$ when the subchannels exactly share a root. Apparently, a *large* value of the performance measure ψ flags a problematic channel. For example, this measure assumes a large value for channel $A(z)$ in [4] which discusses robustness concerns for channel identification using second order statistics of the received signal.

One can also derive useful bounds on σ_e^2:

$$\frac{\gamma}{\gamma + \lambda_{\max}(\mathcal{C}\mathcal{C}^t)} \le \sigma_e^2(f_\delta^\star,\delta) \le \frac{\gamma}{\gamma + \lambda_{\min}(\mathcal{C}\mathcal{C}^t)}. \tag{4}$$

2.3. Factoring the Sylvester matrix

In this section, we will take advantage of the approximation $\hat{\sigma}_e^2$ to develop an understanding of the influence of the subchannel roots on MSE. In T-spaced equalizer design one expects channels with roots close to the unit circle to render high MSEs for fixed SNRs. In the fractionally-spaced case, this translates to nearly common roots close to the unit circle. By deriving an expression for $\hat{\sigma}_e^2$ as a function of the subchannel roots we will make this relationship more precise.

A4' The fractionally-spaced equalizer length **perfectly matches** the length of the channel, i.e., $m = n-1$.

Theorem 1 *Let $\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$ be the roots of $c_a(z)$ and $\{\beta_1,\beta_2,\ldots,\beta_n\}$ be the roots of $c_b(z)$. Assume there are no common subchannel roots, i.e. $\alpha_i \ne \beta_j$, $\forall i,j$ and no multiple subchannel roots, i.e. $\alpha_i \ne \alpha_j$, $\forall i \ne j$. Furthermore, assume that the equalizer has length $m = n-1$ (**A4'**). Then, the (square, invertible) matrix $\mathcal{C}$ can be factored as*

$$\mathcal{C} \quad = \quad \mathcal{U}^{-1}\mathcal{W}\mathcal{V}$$

where

$$\mathcal{U} \quad = \quad \left(\begin{array}{ccccc} 1 & \beta_1^{-1} & \beta_1^{-2} & \ldots & \beta_1^{-p} \\ 1 & \beta_2^{-1} & \beta_2^{-2} & \ldots & \beta_2^{-p} \\ & & \vdots & & \\ 1 & \beta_n^{-1} & \beta_n^{-2} & \ldots & \beta_n^{-p} \\ \hline 1 & \alpha_1^{-1} & \alpha_1^{-2} & \ldots & \alpha_1^{-p} \\ 1 & \alpha_2^{-1} & \alpha_2^{-2} & \ldots & \alpha_2^{-p} \\ & & \vdots & & \\ 1 & \alpha_n^{-1} & \alpha_n^{-2} & \ldots & \alpha_n^{-p} \end{array}\right),$$

$$\mathcal{V} = \begin{pmatrix} \mathcal{V}_\beta & 0 \\ 0 & \mathcal{V}_\alpha \end{pmatrix}, \quad \mathcal{W} = \begin{pmatrix} \mathcal{W}_\beta & 0 \\ 0 & \mathcal{W}_\alpha \end{pmatrix}$$

with

$$\mathcal{V}_\beta = \begin{pmatrix} 1 & \beta_1^{-1} & \beta_1^{-2} & \ldots & \beta_1^{-n+1} \\ 1 & \beta_2^{-1} & \beta_2^{-2} & \ldots & \beta_2^{-n+1} \\ & & \vdots & & \\ 1 & \beta_n^{-1} & \beta_n^{-2} & \ldots & \beta_n^{-n+1} \end{pmatrix},$$

$$\mathcal{W}_\beta = \begin{pmatrix} c_a(\beta_1) & & & \\ & c_a(\beta_2) & & \\ & & \ddots & \\ & & & c_a(\beta_n) \end{pmatrix}$$

($\mathcal{V}_\alpha, \mathcal{W}_\alpha$ similarly defined), and has determinant

$$\det(\mathcal{C}) \quad = \quad \prod_i c_a(\beta_i) \prod_i c_b(\alpha_i) \prod_{i,j} (\alpha_i - \beta_j)^{-1}.$$

Using the theorem above (due to length restrictions the proof is not included[5]) and the fact that $\mathcal{C}$ is invertible we observe that

$$\hat{\sigma}_e^2(\delta) = \gamma \ \left[(\mathcal{C}^{-1})^t \mathcal{C}^{-1}\right]_{\delta,\delta} = \gamma \ \left[(\mathcal{F}^\dagger)^t \mathcal{F}^\dagger\right]_{\delta,\delta} = \gamma \ ||f_\delta^\dagger||^2$$

where $\mathcal{F}^\dagger$ is a matrix composed of column vectors $f_\delta^\dagger$. Each vector $f_\delta^\dagger$ is a **zero-forcing** FSE such that

$$\mathcal{C} f_\delta^\dagger = t_\delta \quad \text{for} \quad 0 \le \delta \le 2n-1.$$

Hence, to find an approximation to the MSE achieved by $f_\delta^\star$, we must compute the **noise gain**, or squared norm, of the zero forcing solution $f_\delta^\dagger$.

Due to the invertibility of $\mathcal{U}$ and $\mathcal{W}$, solving the equation $\mathcal{C}f = t_\delta$ for f (with solution $f = f_\delta^\dagger$) is equivalent to solving $\mathcal{V}f = \mathcal{W}^{-1}\mathcal{U}t_\delta$, which can be viewed as a **polynomial interpolation problem** [3]. This problem is solved using the **Lagrange Interpolation Formula**, which gives

$$f_{a,\delta}^\dagger(z) \quad = \quad \sum_{i=1}^n \frac{\beta_i^{-\delta-1}}{c_a(\beta_i)c_b'(\beta_i)} \frac{c_b(z)}{(1-\beta_i z^{-1})}$$

$$f_{b,\delta}^\dagger(z) \quad = \quad \sum_{i=1}^n \frac{\alpha_i^{-\delta-1}}{c_b(\alpha_i)c_a'(\alpha_i)} \frac{c_a(z)}{(1-\alpha_i z^{-1})}.$$

where $c_b'(\beta_i) = \frac{d}{dz} c_b(z)|_{z=\beta_i}$, and likewise for $c_a'(\alpha_i)$.

Thus we have derived the perfect length, zero forcing FSE solution in subfilter polynomial form. The next step is to compute its noise gain, which can be expressed as $||f_\delta^\dagger||^2 = ||f_{a,\delta}^\dagger||^2 + ||f_{b,\delta}^\dagger||^2$. This can be crudely approximated by $\hat{\sigma}_e^2 \approx \check{\sigma}_e^2$, with

$$\check{\sigma}_e^2 = \gamma ||c_b||^2 \left| \sum_{i=0}^n \frac{\beta_i^{-\delta-1}}{c_a(\beta_i)c_b'(\beta_i)} \right|^2 + \gamma ||c_a||^2 \left| \sum_{i=0}^n \frac{\alpha_i^{-\delta-1}}{c_a(\alpha_i)c_b'(\alpha_i)} \right|^2.$$

2.4. Example

A simple example will serve to illustrate the approximations $\hat{\sigma}_e^2$ and $\check{\sigma}_e^2$. For the $T/2$-spaced channel in Figure 2 the true MSE and the approximations are plotted as functions of delay for SNRs of 30 dB and 60 dB in Figure 3. Notice that the channel is mixed phase; $\hat{\sigma}_e^2, \check{\sigma}_e^2$ are proportional to weighted sums of the subchannel roots raised to δ, indicating that the best MSE delay should be somewhere in the middle of the overall channel-FSE impulse response. Also observe that as $\gamma \to 0$, $\sigma_e^2 = \hat{\sigma}_e^2$; at 60 dB, for instance, the curves for the true MSE and $\hat{\sigma}_e^2$ are very close. It should be clear that $\check{\sigma}_e^2$ is only good for showing a trend in MSE as a function of delay for a particular channel.

[5]The proof is available on request to the 1^{st} author at *raulc@ee.cornell.edu*

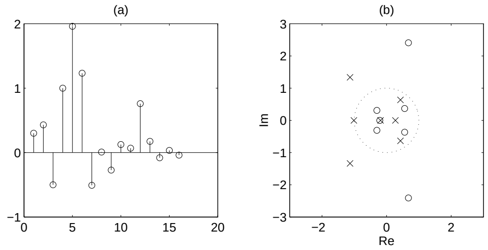

Figure 2. (a) $T/2$-spaced channel impulse response c, and (b) roots of subchannels c_a ($\circ$) and c_b ($\times$).

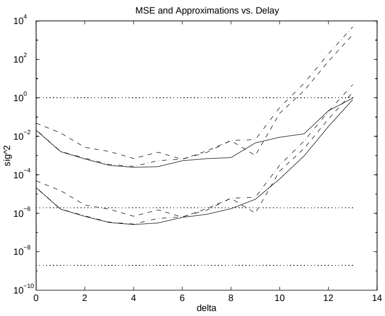

Figure 3. MSE and approximations versus δ for 30 and 60 dB SNR: true MSE σ_e^2 (solid), noise gain approximation $\hat{\sigma}_e^2$ (dash-dot), and crude approximation $\check{\sigma}_e^2$ (dashed). Upper and lower bounds (4) (dotted).

3. IIR MMSE-FSE DESIGN

In this section we derive the MMSE-FSE without constraining the equalizer to be FIR. Aside from the MMSE-FSE calculation, we extract information from this result to classify a channel's severity with respect to the T/2 FIR equalizer. The proposed classification is justified through the application of **Frame Theory** [2] to **Filter Banks** [1], of which Figure 1 is a special case.

3.1. Filter bank structure

In deriving the MMSE-FSE, we do not restrict the structure to be FIR (**A4**), nor the matched filter to be a component forced on the solution [6].

A4" The fractionally-spaced equalizer can be IIR.

Theorem 2 *The MMSE-FSE is given by the IIR subfilters*[6]

$$f_a^\star(z) = \frac{\tilde{c}_a(z)}{Q(z)+\gamma}, \quad f_b^\star(z) = \frac{\tilde{c}_b(z)}{Q(z)+\gamma} \tag{5}$$

$\gamma \geq 0$ with MSE

$$\sigma_e^2(f^\star) \quad = \quad \frac{1}{2\pi} \int_0^{2\pi} \frac{\gamma}{Q(\omega)+\gamma} d\omega \tag{6}$$

where $Q(z) = c_a(z)\tilde{c}_a(z) + c_b(z)\tilde{c}_b(z)$. For $\gamma = 0$, the additional requirement that $Q(\omega) \neq 0 \ \forall \omega$ is needed[5].

[6]Notation: for a real polynomial $p(z)$, the adjoint $\tilde{p}(z) = p(z^{-1})$ and $p(\omega) = p(e^{j\omega})$.

Observe the simple structure that arises in the solution: each subfilter in (5) is composed of a T-spaced matched filter (i.e., $\tilde{c}_a, \tilde{c}_b$) and a T-spaced whitening filter (i.e., $\frac{1}{Q+\gamma}$).

Let $A_c = \min_\omega Q(\omega) + \gamma, \quad B_c = \max_\omega Q(\omega) + \gamma$. Using (6) and A_c, B_c we can redefine MSE bounds for the IIR case

$$\frac{\gamma}{B_c} \leq \sigma_e^2(f^\star) \leq \frac{\gamma}{A_c}.$$

3.2. The matched filter MSE bound

Let the truncated series

$$W^{(j)}(z) = \frac{2}{A_c + B_c}\sum_{i=0}^{j} r^i, \text{ with } r = 1 - \frac{2(Q(z)+\gamma)}{A_c + B_c} \quad (7)$$

be the j^{th} order approximation of the whitening filter $\frac{1}{Q+\gamma}$. This follows from the application of Frame Theory [2] to Filter Banks [1], and its extension to the noise-present case. Now consider a FSE $f^{(j)}$ which uses matched filters $\tilde{c}_a, \tilde{c}_b$, but replaces the whitening filter $\frac{1}{Q+\gamma}$ with $W^{(j)}$. We compare the MSE achieved by $f^{(j)}$ to the MMSE via the difference $\sigma_e^2(f^{(j)}) - \sigma_e^2(f^\star)$ as an indicator of how MSE is mitigated by increasing FSE length.

For the first order approximation ($j = 0$) this difference is bounded by

$$\sigma_e^2(f^{(0)}) - \sigma_e^2(f^\star) \leq r_{\max} \equiv \frac{B_c/A_c - 1}{B_c/A_c + 1}. \quad (8)$$

which is the expression also obtained for a noiseless frame operator in [2]. We call $\sigma_e^2(f^{(0)})$ the matched filter MSE (MF-MSE) bound, since it is in fact an upper bound on the MSE achieved by any FSE longer than the channel.

Utilizing the bounds A_c, B_c, we can produce a bound on the MSE increase for any j as

$$\sigma_e^2(f^{(j)}) - \sigma_e^2(f^\star) \quad \leq \quad r_{\max}^2\left((B_c/A_c)(1 - r_{\max}^j) + 1\right)^2$$

This result suggests correlation between the MSE and B_c/A_c for a fixed MSE performance. That is, an increase in B_c/A_c may correspond to an increase in the length of $W^{(j)}$.

3.3. Channel classification via MF-MSE bound

In effect, B_c/A_c categorizes the channel severity in terms of the solution synthesis ability of MMSE-FSE. The proposed bounds are valid for an IIR equalizer, but can be applied to FIR equalizers, calculated via (2) when the SNR is low enough. Empirically, we have determined the validity of the MF-MSE bound/indicator applied to FIR MMSE-FSE for the channels in Table 1 when the SNR is 47 dB or less. Within this range, it is observed that the calculation $f_a\tilde{f}_a + f_b\tilde{f}_b$ obeys the property of its maximum and minimum falling between $1/B_c$ and $1/A_c$ (which is strictly the case for the IIR solution).

Table 1 lists 13 terrestrial microwave channels (from the **Applied Signal Technology** database available at *http://spib.rice.edu/spib.html*) classified using the MF-MSE bound.

Label	$r_{\max}$	Actual MF-MSE (dB)	MF-MSE Bound (dB)	MMSE 350 taps	MMSE 100 taps
chan1	0.34	-12.43	-4.68	-46.7	-39.4
chan2	0.56	-7.76	-2.51	-46.3	-42.3
chan4	0.84	-6.25	-0.78	-44.8	-28.8
chan5	0.62	-9.95	-2.11	-46.5	-37.7
chan6	0.33	-12.82	-4.80	-46.8	-41.1
chan8	0.44	-12.00	-3.52	-46.7	-37.2
chan9	0.85	-5.77	-0.69	-44.5	-41.9
chan10	0.88	-4.09	-0.57	-44.1	-36.0
chan11	0.77	-7.84	-1.14	-45.7	-33.3
chan12	0.24	-17.31	-6.20	-46.9	-40.7
chan13	0.31	-14.70	-5.02	-46.8	-44.9

Table 1. Microwave channel classification and measurements with regards to the MF-MSE Bound. Tests performed at SNR=47 dB. Channel lengths vary between 200 and 300 $T/2$-spaced taps.

4. CONCLUSION

This paper has provided solutions for both FIR (under **A4'**) and IIR MMSE-FSE designs as well as expressions and approximations to the achieved MSE. These results, expressed in terms of subchannel parameters, roots and spectra can be used as analytical tools for understanding the various open issues involved in MMSE-FSE design.

To complete the theory on MMSE-FSE design, however, additional work remains. Some unresolved items follow:

- Expressions and bounds for MSE when the equalizer is *shorter* than the channel as function of the channel parameters, SNR, and equalizer length. Ultimately, a simple scheme for deriving FSE length for a particular channel, SNR and a target SER.
- The best suited method for modeling channels as autoregressive moving average (ARMA) filters from empirical data.
- Simple design rule for determining optimum MSE delay δ from channel parameters and equalizer length.

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