J_{CM} and $\nabla_{\mathbf{f}}(J_{\mathrm{CM}})$: Three Important Cases*

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1 Introduction

The purpose of this document is to present versions of the Godard 2-2 cost function for some important cases[1] and to accompany these cost functions with expressions for the gradient in equalizer space. The gradient is necessary for constant-modulus error gradient-descent algorithms, as included in The BERGulator. We restrict our attention to three scenarios:

1. PAM source, real-valued channel response, and real-valued Gaussian noise
2. PAM source and rotationally-invariant complex-valued Gaussian noise
3. QAM source and rotationally-invariant complex-valued Gaussian noise

with the additional assumption that the source has unit variance. The first scenario above is academically motivated, though is feasible in practice if the data transmission is implemented at baseband. The second two scenarios are applicable to a communication system employing modulation. In general, this latter situation implies a complex-valued channel response; a real-valued channel response would imply that the respective frequency response is conjugate-symmetric around the carrier frequency, which is not true in general.

The notation employed is as follows: the system impulse response is $\mathbf{h} = [h_0, \ldots, h_{P-1}]^T$, the equalizer coefficients are $\mathbf{f} = [f_0, \ldots, f_{2N-1}]^T$, and the $(P \times 2N)$ channel convolution matrix is $\mathbf{C}$. Hence, $\mathbf{h} = \mathbf{C}\mathbf{f}$. Finally, it will be convenient to denote the i^{th} column of $\mathbf{C}^T$ by $\mathbf{c}_i$, since then $h_i = \mathbf{c}_i^T \mathbf{f}$.

2 General Godard 2-2 Cost

In terms of the equalizer output $\{y_n\}$, the Godard 2-2 cost is defined as

$$J_{CM} \stackrel{\triangle}{=} E\left[\left(|y_n|^2 - \kappa_s \sigma_s^2\right)^2\right], \tag{1}$$

where σ_s^2 and κ_s denote the variance and the normalized kurtosis of the source process $\{s_n\}$, respectively. These quantities are themselves defined as follows:

$$\sigma_s^2 \stackrel{\triangle}{=} E\big[|s_n|^2\big] \tag{2}$$

$$\kappa_s \stackrel{\triangle}{=} \frac{E\big[|s_n|^4\big]}{\big(E\big[|s_n|^2\big]\big)^2}. \tag{3}$$

The analogously-defined quantities σ_n^2 and κ_n pertain to the channel noise process $\{w_n\}$.

*Incorporates corrections to an earlier version, dated June 1997.

[1] This work builds on results that Jim Behm derived while visiting the Cornell BERG.

Equation (4) presents the most general form of the Godard 2-2 cost function. The only assumption made on the source and noise in (4) is that they are i.i.d. and mutually independent.

$$\begin{aligned} J_{CM} &= \kappa_s\sigma_s^4\sum_{i=0}^{P-1}|h_i|^4 + 2\sigma_s^4\sum_{i=0}^{P-1}\sum_{m=0,m\neq i}^{P-1}|h_i|^2|h_m|^2 + |E[s_n^2]|^2\sum_{i=0}^{P-1}\sum_{j=0,j\neq i}^{P-1}h_i^2(h_j^*)^2 \\ &+\kappa_w\sigma_w^4\sum_{i=0}^{2N-1}|f_i|^4 + 2\sigma_w^4\sum_{i=0}^{2N-1}\sum_{m=0,m\neq i}^{2N-1}|f_i|^2|f_m|^2 + |E[w_n^2]|^2\sum_{i=0}^{2N-1}\sum_{j=0,j\neq i}^{2N-1}f_i^2(f_j^*)^2 \\ &+(E[s_n^2]\sum_{i=0}^{P-1}h_i^2)(E[w_n^2]\sum_{i=0}^{2N-1}f_i^2)^* + 4\sigma_s^2\sigma_w^2\|\mathbf{h}\|_2^2\|\mathbf{f}\|_2^2 \\ &+(E[s_n^2]\sum_{i=0}^{P-1}h_i^2)^*(E[w_n^2]\sum_{i=0}^{2N-1}f_i^2) - 2\sigma_s^2\kappa_s(\sigma_s^2\|\mathbf{h}\|_2^2 + \sigma_w^2\|\mathbf{f}\|_2^2) + \sigma_s^4{\kappa_s}^2 \end{aligned} \quad (4)$$

3 PAM, Real-valued Channel, Real-Valued AWGN

Given this special case, we can simplify (4) using the following:

- unit-variance source: $\sigma_s^2 = 1$
- real-valued source: $E[s_n^2] = \sigma_s^2 = 1$
- real-valued noise: $E[w_n^2] = \sigma_w^2$
- Gaussian noise: $\kappa_w = 3$

The resulting CM cost appears in (6).

$$\begin{aligned} J_{CM}|_{\mathbb{R}} &= (\kappa_s - 3)\sum_{i=0}^{P-1}h_i^4 + 3\|\mathbf{h}\|_2^4 + 3\sigma_w^4\|\mathbf{f}\|_2^4 + 6\sigma_w^2\|\mathbf{h}\|_2^2\|\mathbf{f}\|_2^2 \quad (5) \\ &-2\kappa_s(\|\mathbf{h}\|_2^2 + \sigma_w^2\|\mathbf{f}\|_2^2) + {\kappa_s}^2 \end{aligned}$$

Using the identities in Appendix A, the real-valued gradient with respect to $\mathbf{f}$ is

$$\nabla_{\mathbf{f}}(J_{CM}|_{\mathbb{R}}) = 4(\kappa_s - 3)\sum_{i=0}^{P-1}h_i^3\mathbf{c}_i + 4\big(3\|\mathbf{h}\|_2^2 + 3\sigma_w^2\|\mathbf{f}\|_2^2 - \kappa_s\big)\big(\mathbf{C}^T\mathbf{h} + \sigma_w^2\mathbf{f}\big) \quad (6)$$

4 PAM, Rotationally-invariant Complex-Valued AWGN

Given this special case, we can simplify (4) using the following:

- unit-variance source: $\sigma_s^2 = 1$
- real-valued source: $E[s_n^2] = \sigma_s^2 = 1$
- rotationally-invariant noise: $E[w_n^2] = 0$
- complex-valued Gaussian noise: $\kappa_w = 2$

The resulting CM cost appears in (7).

$$\begin{aligned} J_{CM}|_{PAM} &= (\kappa_s - 3)\sum_{i=0}^{P-1}|h_i|^4 + 2\|\mathbf{h}\|_2^4 + \left|\mathbf{h}^T\mathbf{h}\right|^2 + 2\sigma_w^4\|\mathbf{f}\|_2^4 \\ &+4\sigma_w^2\|\mathbf{h}\|_2^2\|\mathbf{f}\|_2^2 - 2\kappa_s(\|\mathbf{h}\|_2^2 + \sigma_w^2\|\mathbf{f}\|_2^2) + {\kappa_s}^2 \end{aligned} \quad (7)$$

Using the identities in Appendix B, the complex-valued gradient with respect to $\mathbf{f}$ is

$$\begin{aligned}\nabla_{\mathbf{f}}(J_{CM}|_{PAM}) &= 4(\kappa_s - 3)\sum_{i=0}^{P-1} |h_i|^2 h_i \mathbf{c}_i^* \\ &+4\mathbf{h}^T\mathbf{h}(\mathbf{C}^T\mathbf{h})^* + 4\big(2\,\|\mathbf{h}\|_2^2 + 2\sigma_w^2\,\|\mathbf{f}\|_2^2 - \kappa_s\big)\big(\mathbf{C}^H\mathbf{h} + \sigma_w^2\mathbf{f}\big) \qquad (8)\end{aligned}$$

5 QAM, Rotationally-invariant Complex-Valued AWGN

Given this special case, we can simplify (4) using the following:

- unit-variance source: $\sigma_s^2 = 1$
- QAM source: $E[s_n^2] = 0$
- rotationally-invariant noise: $E[w_n^2] = 0$
- complex-valued Gaussian noise: $\kappa_w = 2$

The resulting CM cost appears in (9).

$$\begin{aligned}J_{CM}|_{QAM} &= (\kappa_s - 2)\sum_{i=0}^{P-1} |h_i|^4 + 2\,\|\mathbf{h}\|_2^4 + 2\sigma_w^4\,\|\mathbf{f}\|_2^4 + 4\sigma_w^2\,\|\mathbf{h}\|_2^2\,\|\mathbf{f}\|_2^2 \\ &-2\kappa_s(\,\|\mathbf{h}\|_2^2 + \sigma_w^2\,\|\mathbf{f}\|_2^2) + {\kappa_s}^2 \qquad (9)\end{aligned}$$

Using the identities in Appendix B, the complex-valued gradient with respect to $\mathbf{f}$ is

$$\begin{aligned}\nabla_{\mathbf{f}}(J_{CM}|_{QAM}) &= 4(\kappa_s - 2)\sum_{i=0}^{P-1} |h_i|^2 h_i \mathbf{c}_i^* \\ &+4\big(2\,\|\mathbf{h}\|_2^2 + 2\sigma_w^2\,\|\mathbf{f}\|_2^2 - \kappa_s\big)\big(\mathbf{C}^H\mathbf{h} + \sigma_w^2\mathbf{f}\big) \qquad (10)\end{aligned}$$

A Real-valued Gradient Identities

In this appendix, we present some identities used in deriving the gradient expression in Section 3. The real-valued gradient $\nabla_{\mathbf{f}}(\cdot)$ is defined as the standard vector-valued derivative

$$\nabla_{\mathbf{f}}(\cdot) \triangleq \frac{\partial}{\partial \mathbf{f}}\Big(\cdot\Big) = \left[\frac{\partial}{\partial f_o}\Big(\cdot\Big), \cdots, \frac{\partial}{\partial f_{2N-1}}\Big(\cdot\Big)\right]^T$$

where $\mathbf{f}$ is real-valued. The identities follow:

$$\nabla_{\mathbf{f}}\big(\|\mathbf{f}\|_2^2\big) = 2\mathbf{f} \tag{11}$$

$$\nabla_{\mathbf{f}}\big(\|\mathbf{f}\|_2^4\big) = 4\,\|\mathbf{f}\|_2^2\,\mathbf{f} \tag{12}$$

$$\nabla_{\mathbf{f}}\big(\|\mathbf{h}\|_2^2\big) = 2\mathbf{C}^H\mathbf{h} \tag{13}$$

$$\nabla_{\mathbf{f}}\big(\|\mathbf{h}\|_2^4\big) = 4\,\|\mathbf{h}\|_2^2\,\mathbf{C}^H\mathbf{h} \tag{14}$$

$$\nabla_{\mathbf{f}}\left(\sum_{i=0}^{P-1} h_i^4\right) = 4\sum_{i=0}^{P-1} h_i^3\mathbf{c}_i, \text{ where } \mathbf{c}_i \text{ is the } i^{th} \text{ column of } \mathbf{C}^T \tag{15}$$

B Complex-valued Gradient Identities

In this appendix, we present some identities used in deriving the gradient expressions in Sections 4 and 5. The complex-valued gradient $\nabla_{\mathbf{f}}(\cdot)$ is defined by

$$\nabla_{\mathbf{f}}(\cdot) \triangleq \frac{\partial}{\partial \mathbf{f}_r}\Big(\cdot\Big) + j\frac{\partial}{\partial \mathbf{f}_i}\Big(\cdot\Big)$$

where $\mathbf{f} = \mathbf{f}_r + j\mathbf{f}_i$ and both $\mathbf{f}_r$ and $\mathbf{f}_i$ are real-valued. The identities follow:

$$\nabla_{\mathbf{f}}\big(\|\mathbf{f}\|_2^2\big) = 2\mathbf{f} \tag{16}$$

$$\nabla_{\mathbf{f}}\big(\|\mathbf{f}\|_2^4\big) = 4\,\|\mathbf{f}\|_2^2\,\mathbf{f} \tag{17}$$

$$\nabla_{\mathbf{f}}\left(\sum_{i=0}^{2N-1} |f_i|^4\right) = 4\big[\,|f_0|^2 f_0, \ldots, |f_{2N-1}|^2 f_{2N-1}\big]^T \tag{18}$$

$$\nabla_{\mathbf{f}}\big(\|\mathbf{h}\|_2^2\big) = 2\mathbf{C}^H\mathbf{h} \tag{19}$$

$$\nabla_{\mathbf{f}}\big(\|\mathbf{h}\|_2^4\big) = 4\,\|\mathbf{h}\|_2^2\,\mathbf{C}^H\mathbf{h} \tag{20}$$

$$\nabla_{\mathbf{f}}\left(\sum_{i=0}^{P-1} |h_i|^4\right) = 4\sum_{i=0}^{P-1} |h_i|^2 h_i\mathbf{c}_i^*, \text{ where } \mathbf{c}_i \text{ is the } i^{th} \text{ column of } \mathbf{C}^T \tag{21}$$

$$\nabla_{\mathbf{f}}\Big(\big|\mathbf{h}^T\mathbf{h}\big|^2\Big) = 4\mathbf{h}^T\mathbf{h}(\mathbf{C}^T\mathbf{h})^* \tag{22}$$