# Iterative Frequency-Domain Equalization of Single-Carrier Transmissions over Doubly-Dispersive Channels

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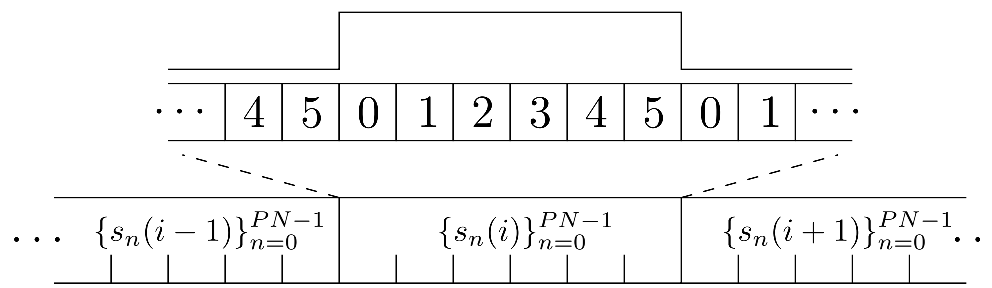
# Background:

- Consider communication over doubly-dispersive channels.
- Options:
  1. Single-Carrier Mod. with Time-Domain Equalization
     - Fast MMSE-DFE: $\mathcal{O}(N_h^2)$ ops/symbol for chan length $N_h$.
     - Low PAPR, no need for guard interval.
  2. Multi-Carrier Mod. with Freq-Domain Equalization
     - Includes ICI mitigation (unlike slow-fading case).
     - $\mathcal{O}(\log N)$ ops/symbol for block length $N$. [Schniter:TSP:04]
     - High PAPR, often requires guard interval!
  3. Single-Carrier Cyclic-Prefix with Freq-Domain Eq
     - $\mathcal{O}(\log N)$ ops/symbol for block length $N$. [Schniter:ASIL:04]
     - Low PAPR, requires guard interval!
- What about FDE for single-carrier modulation *without* guards?

# The PS-FDM Equivalent via Virtual Subcarriers:

- Model input stream as a sequence of $PN$-length frames, and equate each frame with a *rectangularly-windowed cyclic extension* of the time-domain symbols $\{s_n(i)\}_{n=0}^{PN-1}$ in that frame.

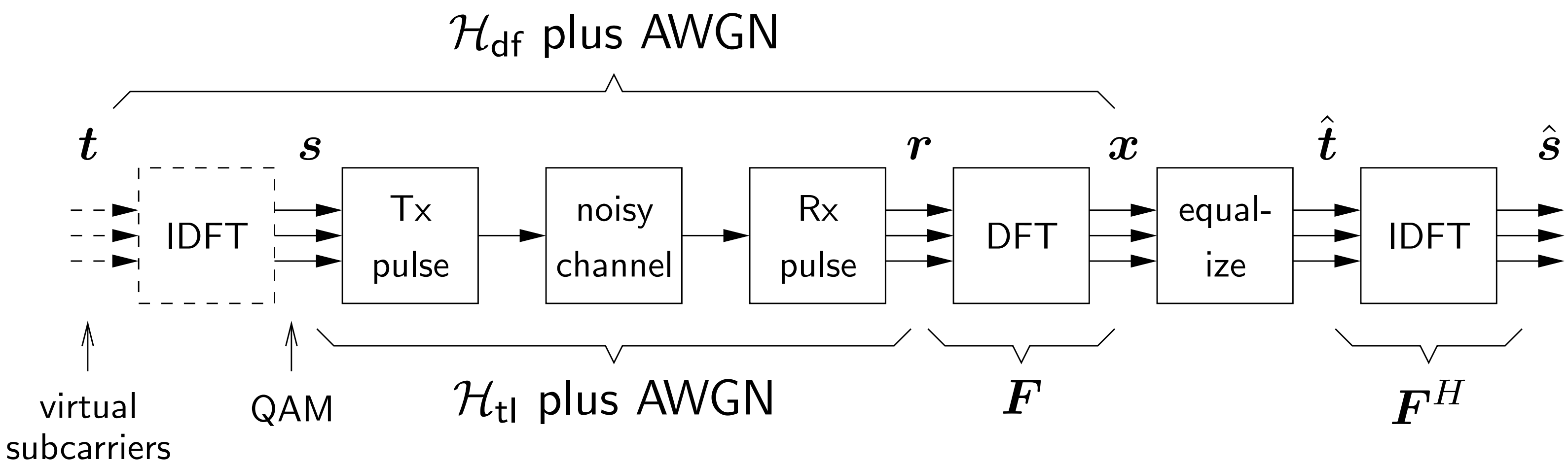



- Now define the *virtual subcarrier* sequence $\{t_k(i)\}_{k=0}^{PN-1}$:

$$t_k(i) = \frac{1}{\sqrt{PN}} \sum_{n=0}^{PN-1} s_n(i) e^{-j\frac{2\pi}{N}kn}$$

- Thus, the single carrier Tx is equivalent to a *pulse-shaped frequency division multiplexing (PS-FDM)* Tx that communicates $\{t_k(i)\}_{k=0}^{PN-1}$ using a rectangular pulse and $0$-length prefix.

## System Model:




$$\boldsymbol{r} = \mathcal{H}_{\mathsf{tl}}\boldsymbol{s} + \boldsymbol{\varepsilon} \qquad (\boldsymbol{\varepsilon} \text{ contains noise and ISI})$$

$$\boldsymbol{x} = \underbrace{\boldsymbol{F}\mathcal{H}_{\mathsf{tl}}\boldsymbol{F}^H}_{\mathcal{H}_{\mathsf{df}}} \underbrace{\boldsymbol{F}\boldsymbol{s}}_{\boldsymbol{t}} + \underbrace{\boldsymbol{F}\boldsymbol{\varepsilon}}_{\boldsymbol{w}}$$

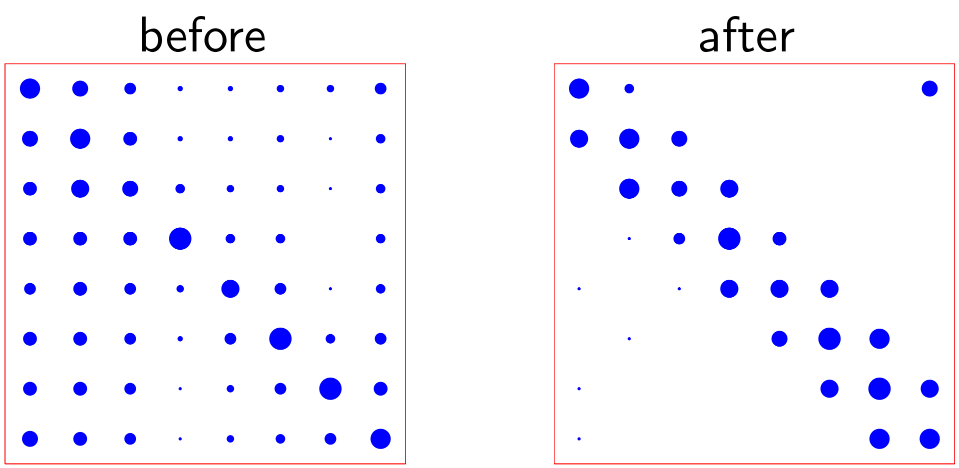
where $\mathcal{H}_{\mathsf{tl}}$ = pulse-shaped circular-convolution matrix,

$\mathcal{H}_{\mathsf{df}}$ = "virtual-subcarrier" coupling matrix.

$\rightsquigarrow$ $\mathcal{H}_{\mathsf{df}}$ diagonal iff channel is LTI and prefix-length is adequate.

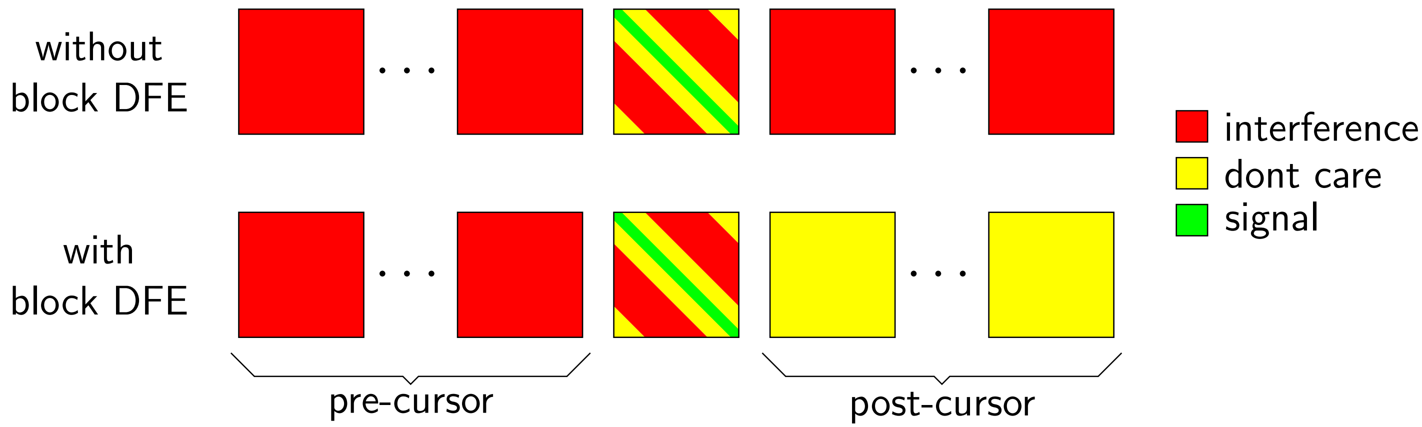
## Receiver Pulse-Shaping:

- Objective:
  - Want to make $\mathcal{H}_{\mathrm{df}}$ sparse for low-complexity detection.
  - Interpretation: virtual-subcarrier ICI-response "shortening".
  - Reminiscent of ISI-shortening for single-carrier MLSD.
- Recall time-domain windowing = Doppler-domain convolution!



# Pulse Design:

- View multicarrier response as a MIMO ISI channel:

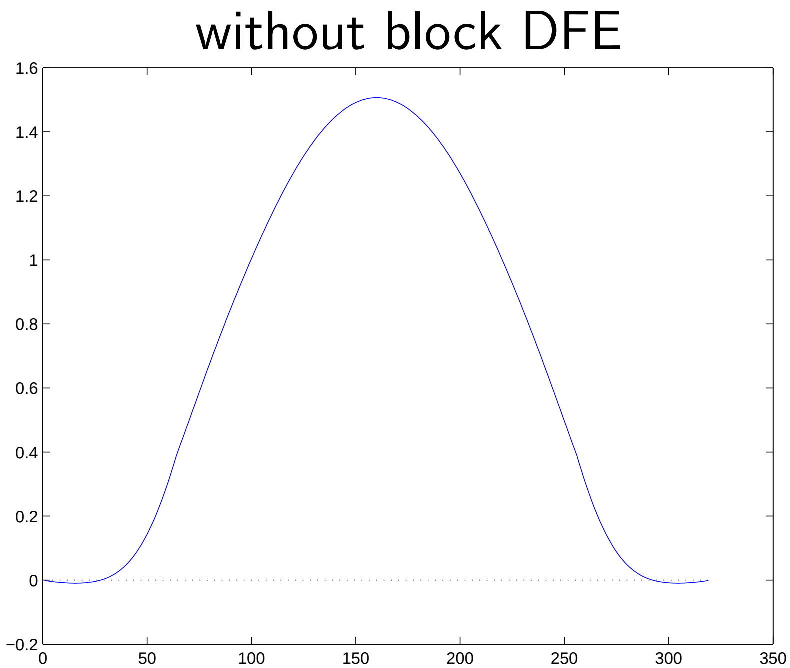


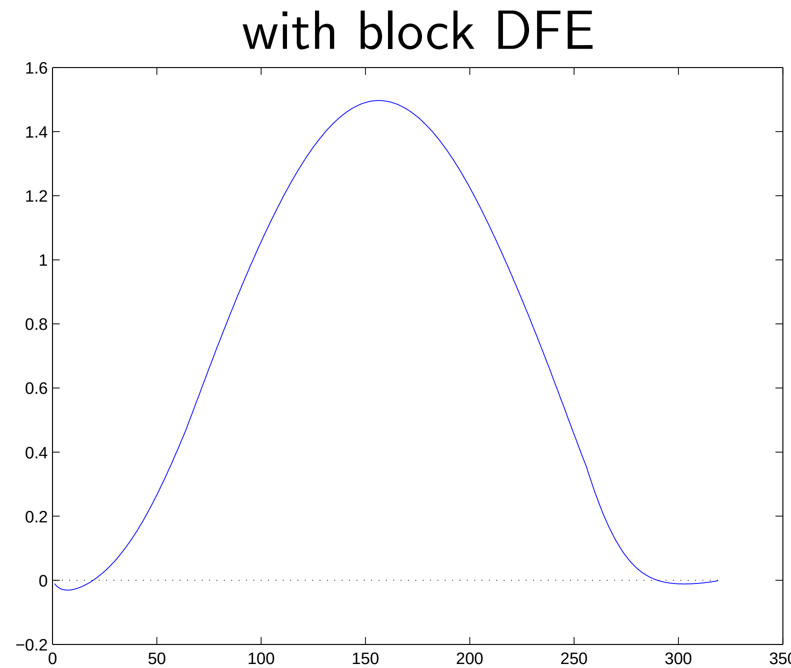

- Max-SINR window coefs found via generalized eigenvector problem:

$$\boldsymbol{b}_\star = \arg\max_{\|\boldsymbol{b}\|^2=N_s} \frac{\boldsymbol{b}^H \boldsymbol{Q}_{\mathsf{s}}(\boldsymbol{a})\boldsymbol{b}}{\boldsymbol{b}^H \boldsymbol{Q}_{\mathsf{ni}}(\boldsymbol{a})\boldsymbol{b}}$$

where $\boldsymbol{a}$ is the $PN$-length rectangular pulse sequence and $\{\boldsymbol{Q}_{\mathsf{s}}, \boldsymbol{Q}_{\mathsf{ni}}\}$ depend on channel *statistics* (e.g., Doppler, delay spread, SNR).

# Example Rx Pulses:

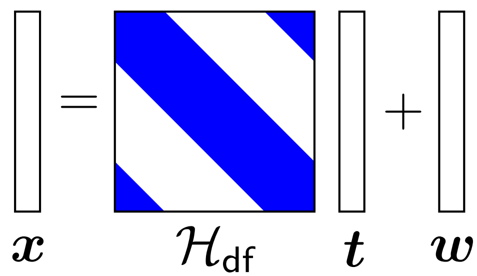



$f_{\mathsf{d}}T_s = 0.01,\ N_h = 64,\ PN = 256,$ SNR$= 10$dB

## Frequency-Domain Equalization:

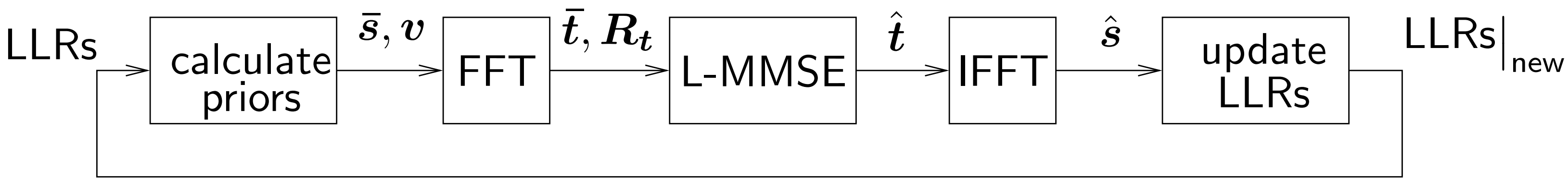
- For now, decouple equalization from decoding (for simplicity).
- With successful pulse design, system model becomes



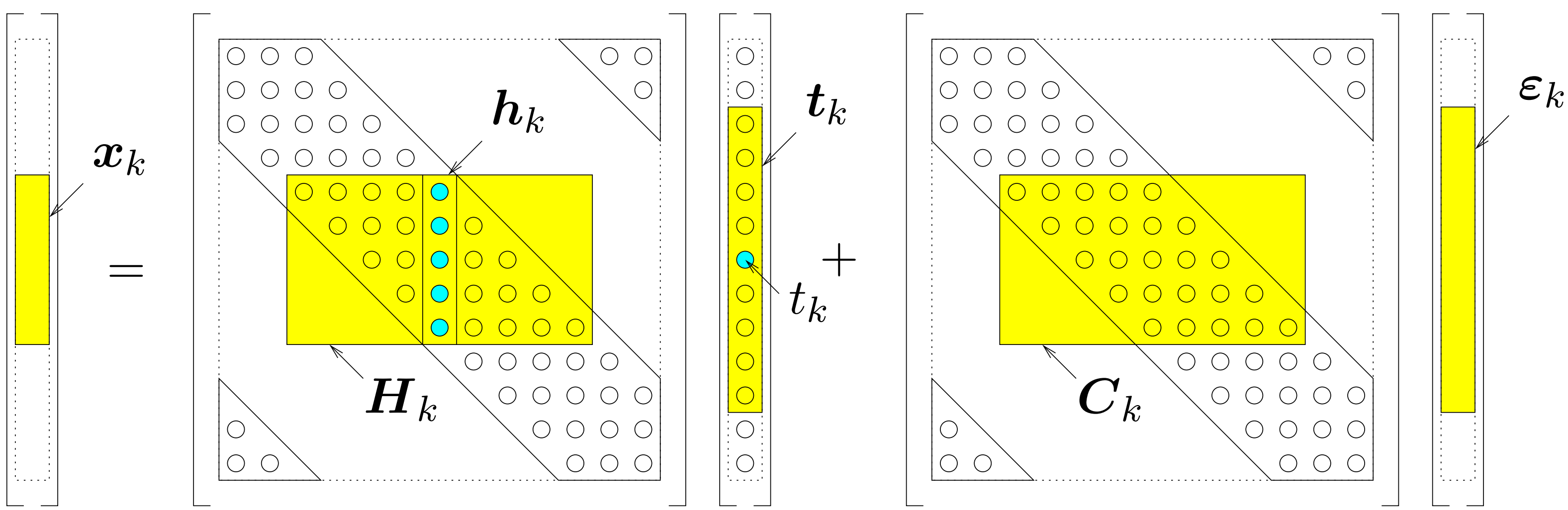

  - $\mathcal{H}_{\mathsf{df}}$ has a banded structure,
  - $\boldsymbol{w}$ is dominated by freq-domain noise.
  - $\boldsymbol{s}$ and $\boldsymbol{t}$ are related through the DFT.
- Equalization strategy leverages three essential properties:
  1. Banded structure of $\mathcal{H}_{\mathsf{df}}$,
  2. Fast algorithm for DFT (i.e., the FFT),
  3. Finite alphabet property of $\boldsymbol{s}$.

# Iterative MMSE Estimation:

*Block Iteration*:




*L-MMSE step for each $k$*:

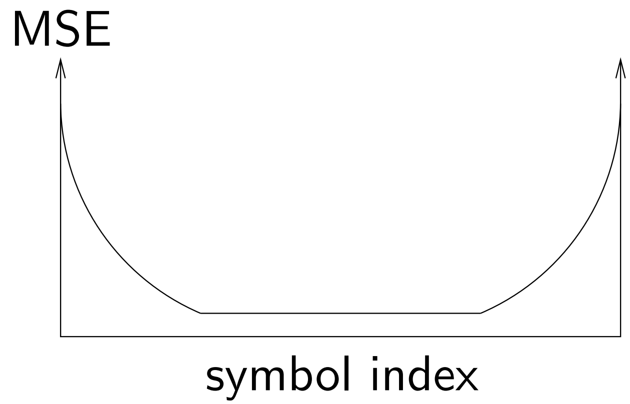


# Algorithm requiring $\mathcal{O}(D^2 \log N)$ operations/symbol:

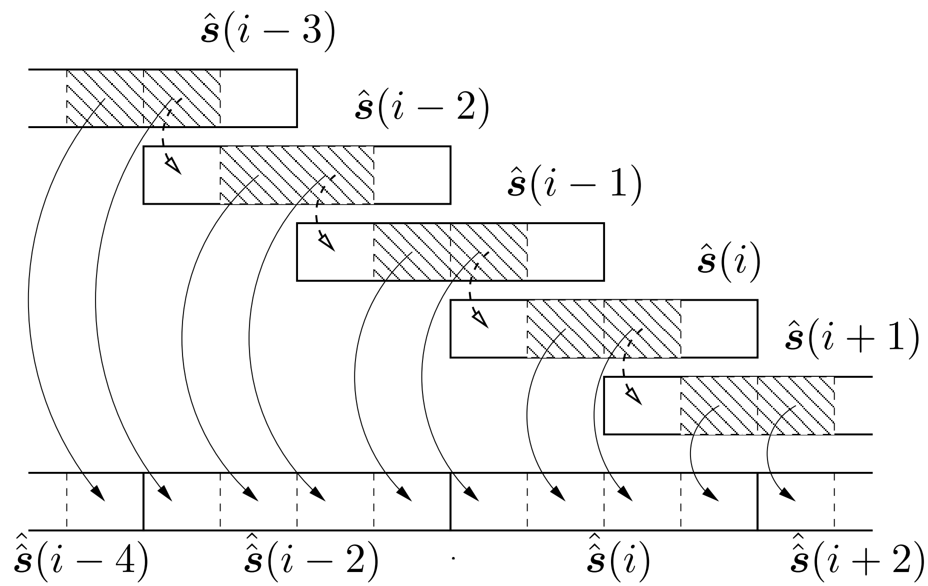
$$
\begin{array}{l}
L^{(0)}(s_k) = 0 \ \forall k \\
\texttt{for}\ i = 0 \ldots, \\
\quad \texttt{for}\ k = 0 \ldots N-1, \\
\quad\quad \bar{s}_k^{(i+1)} = \tanh(L^{(i+1)}(s_k)/2) \\
\quad\quad v_k^{(i+1)} = 1 - (\bar{s}_k^{(i+1)})^2 \\
\quad \texttt{end} \\
\quad \bar{\boldsymbol{t}}^{(i)} = \boldsymbol{F}\bar{\boldsymbol{s}}^{(i)} \\
\quad \texttt{for}\ k = 0 \ldots N-1, \\
\quad\quad \boldsymbol{g}_k^{(i)} = \big(\breve{\mathcal{H}}_k \boldsymbol{F}\,\mathcal{D}(\boldsymbol{v}^{(i)})\boldsymbol{F}^H \breve{\mathcal{H}}_k^H + \sigma^2 \boldsymbol{C}_k \boldsymbol{C}_k^H\big)^{-1} \breve{\mathcal{H}}_k \boldsymbol{F}\,\mathcal{D}(\boldsymbol{v}^{(i)})\boldsymbol{F}^H \boldsymbol{i}_k \\
\quad\quad \hat{t}_k^{(i)} = \bar{t}_k^{(i)} + \boldsymbol{g}_k^{(i)H}(\boldsymbol{x}_k - \breve{\mathcal{H}}_k \bar{\boldsymbol{t}}^{(i)}) \\
\quad \texttt{end} \\
\quad \boldsymbol{Q}^{(i)} = \boldsymbol{F}^H \Big(\sum_{k=0}^{N-1} \breve{\mathcal{H}}_k^H \boldsymbol{g}_k^{(i)} \boldsymbol{i}_k^H\Big)\boldsymbol{F} \\
\quad \boldsymbol{P}^{(i)} = \boldsymbol{F}^H \Big(\sum_{k=0}^{N-1} \boldsymbol{C}_k^H \boldsymbol{g}_k^{(i)} \boldsymbol{i}_k^H\Big)\boldsymbol{F} \\
\quad \hat{\boldsymbol{s}}^{(i)} = \boldsymbol{F}^H \hat{\boldsymbol{t}}^{(i)} \\
\quad \texttt{for}\ k = 0 \ldots N-1, \\
\quad\quad L^{(i+1)}(s_k) = L^{(i)}(s_k) + 4\, \dfrac{\mathrm{Re}\{Q_{k,k}^{(i)}(\hat{s}_k^{(i)} - \bar{s}_k^{(i)})\} + |Q_{k,k}^{(i)}|^2 \bar{s}_k^{(i)}}{\boldsymbol{q}_k^{(i)H}\,\mathcal{D}(\boldsymbol{v}^{(i)})\boldsymbol{q}_k^{(i)} - |Q_{k,k}^{(i)}|^2 v_k^{(i)} + \sigma^2 \|\boldsymbol{p}_k^{(i)}\|^2} \\
\quad \texttt{end} \\
\texttt{end}
\end{array}
$$

# The Need for Frame Overlap:

Windowing causes uneven distribution of symbol errors across frame:




The mid-frame symbol estimates are saved as "final" estimates and their LLRs are used to initialize the next frame. Frames overlap so that all symbols can be reliably estimated.



# Simulation Details:

Channel/Modulator:

- WSSUS Rayleigh fading, uniform delay profile, length $N_h = 64$.
- Uncoded BPSK.

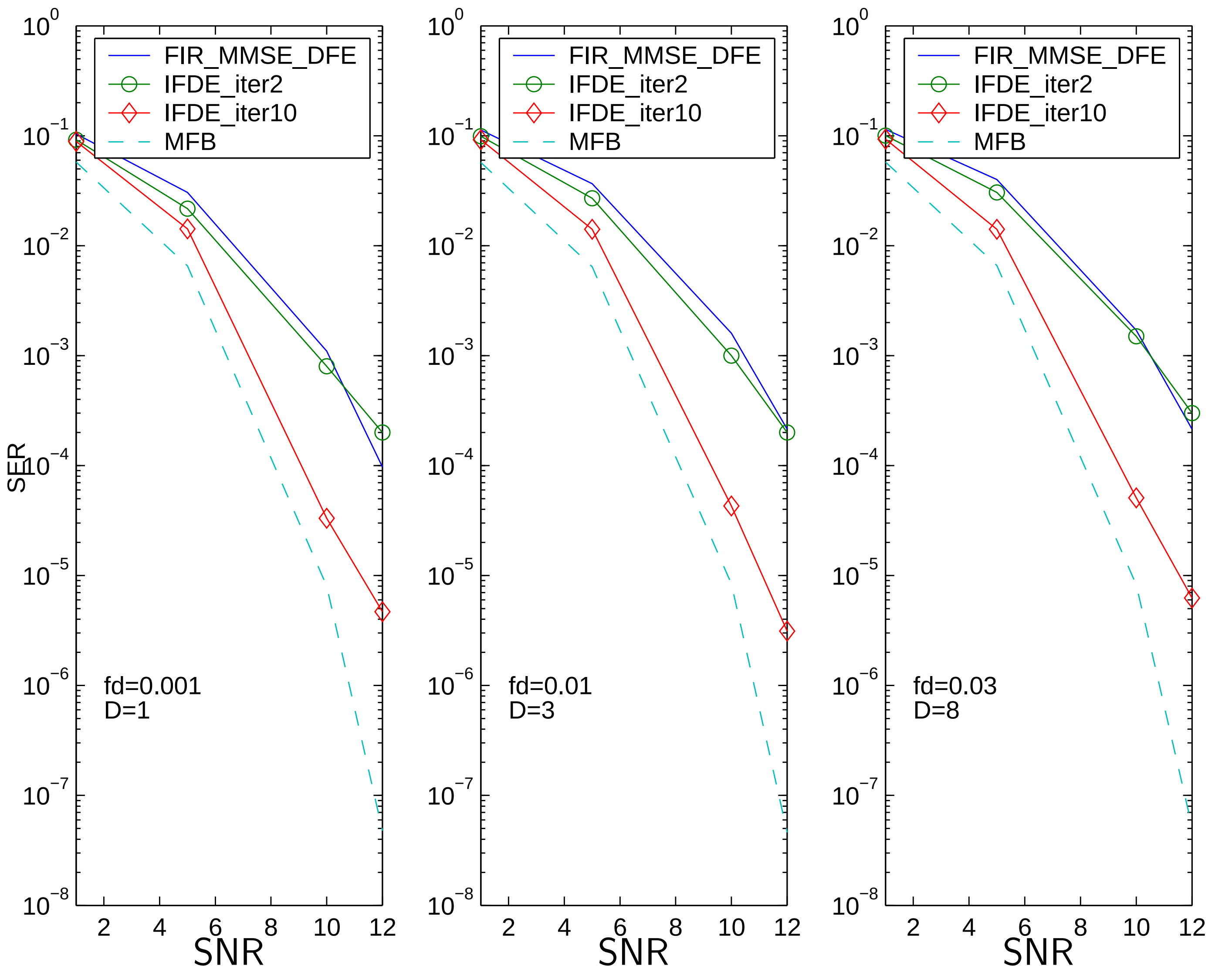
Receiver:

- Frame length $PN = 4N_h$, frame overlap factor $P = 2$.
- ICI radius $D = \lceil f_{\mathsf{d}} T_s PN \rceil$.
- 10 iterations.

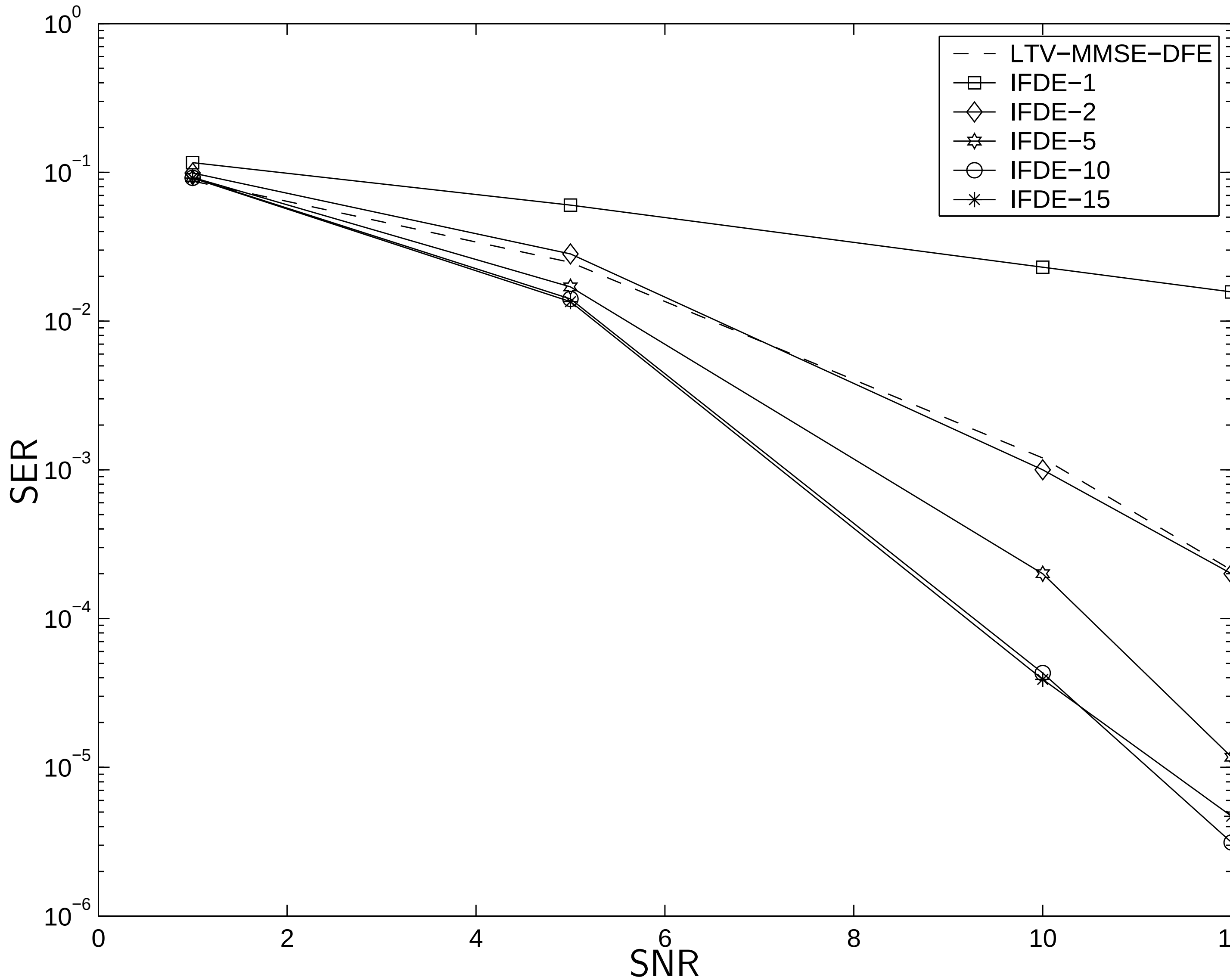
Reference:

- LTV-MMSE-DFE: Update rate $\frac{1}{T_s}$, $\mathcal{O}(N_h^2)$ ops/symbol.
- LTI-MMSE-DFE: Update rate $\frac{1}{N_h T_s}$, $\mathcal{O}(N_h)$ ops/symbol.

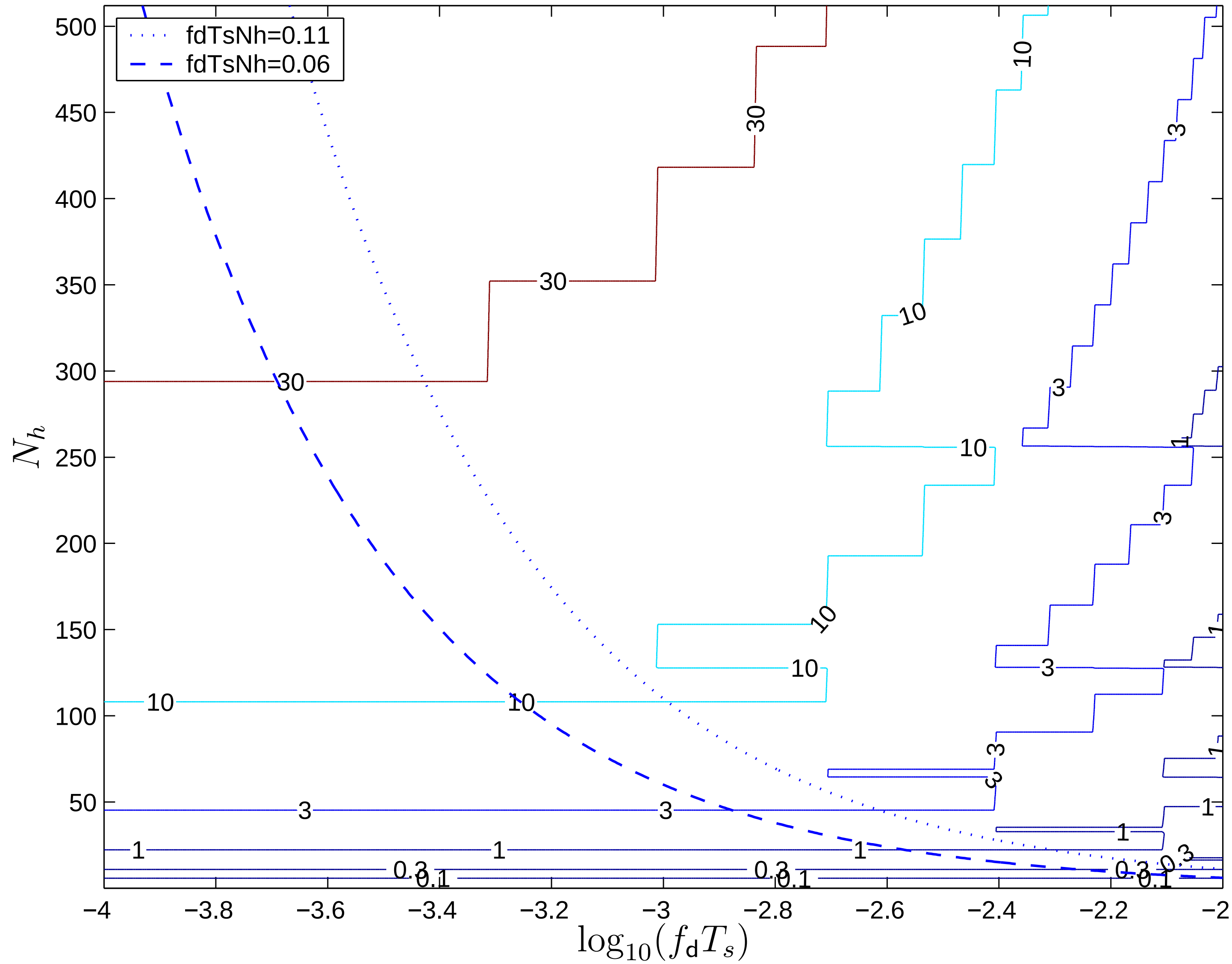
# Uncoded-SER :: FIR-MMSE-DFE Comparison:



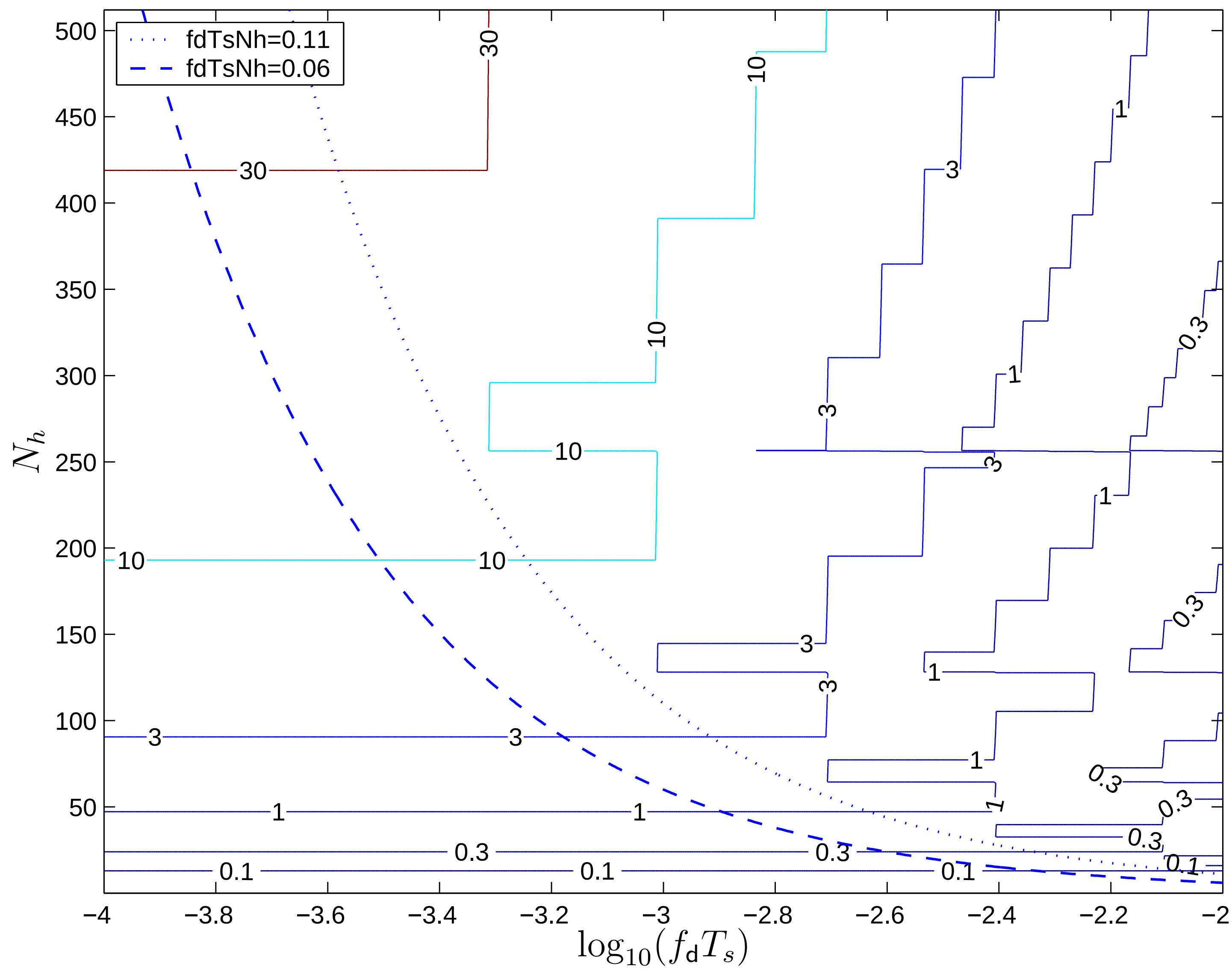
# Uncoded-SER :: Effect of Number of Iterations:



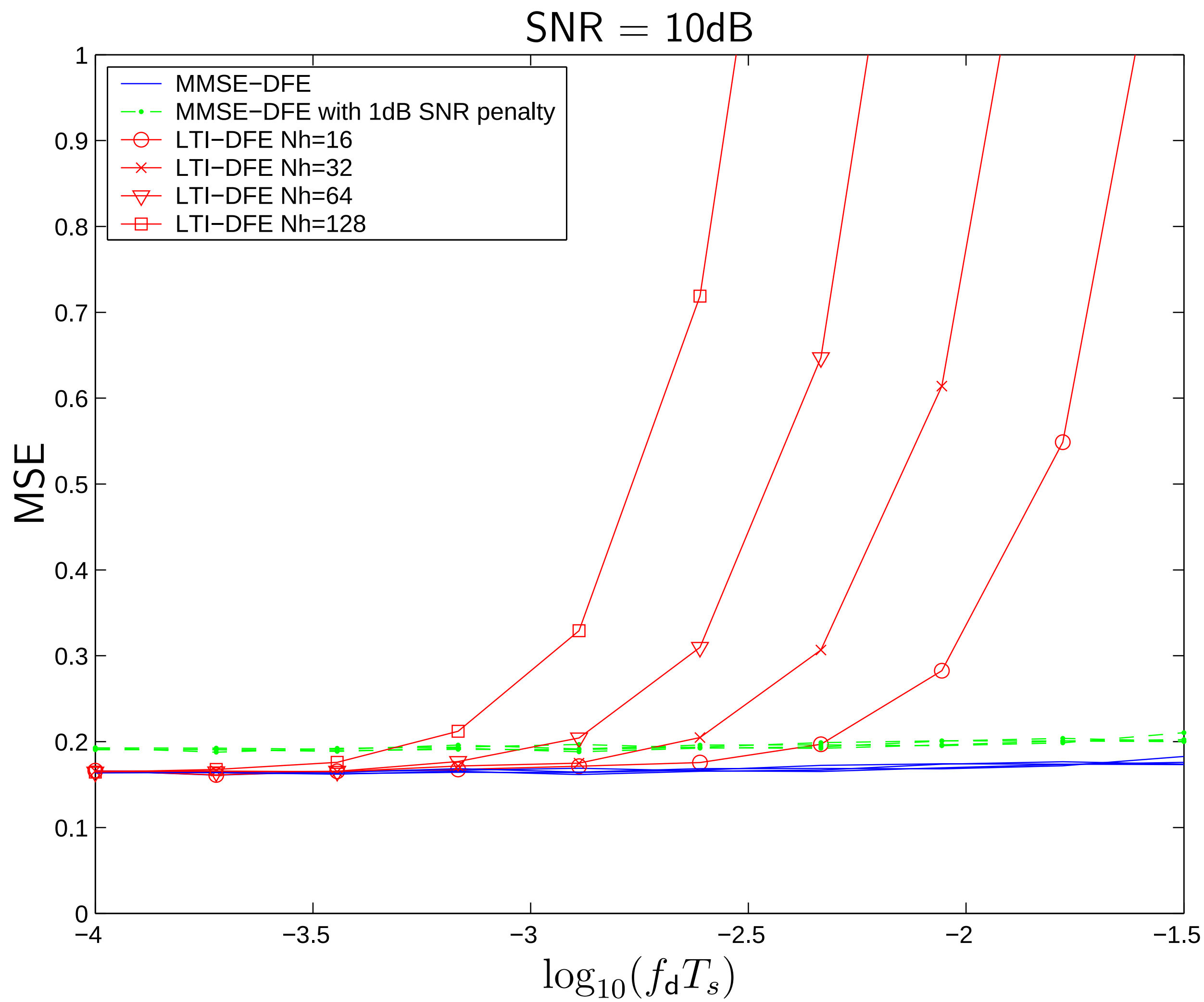
# Complexity Advantage Over LTV-DFE ($M = 2$):



# Complexity Advantage Over LTV-DFE ($M = 10$):



# When does the LTI-DFE break down?




$\rightsquigarrow$ Roughly speaking, when $f_d T_s \cdot N_h > 0.06$

## Summary:

- Freq-domain equalization in doubly-selective channels must deal with ICI as well as ISI.
- We proposed a two-stage frequency-domain equalizer:
  1. SINR-optimal windowing for ICI-response shortening,
  2. Iterative MMSE estimation leveraging finite alphabet and FFT.
- Complexity $\mathcal{O}(\log N)$ ops/symbol, similar to classical frequency-domain equalization approaches (e.g., OFDM).
- Performance equal to LTV-MMSE-DFE at 2 iterations, though much less complex.
- Performance far beyond LTV-MMSE-DFE after 10 iterations, and also less complex when $f_{\mathrm{d}} T_s < 0.006$.
- Soft decoding can be easily incorporated.