CMA-FSE Error Surface Examples

Technical Report

P. Schniter, R. A. Casas, F. Lopez de Victoria

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Abstract

This report shows through simple examples important characteristics of the behavior of CMA applied to fractionally spaced equalization. Specifically, it demonstrates that in the presence of noise or when perfect equalizability is not attainable (due to length restrictions on the equalizer), the CMA-FSE cost function is multi-modal and contains false minima. This observation indicates that proper initialization is necessary for CMA-FSE to converge to the optimal solution. This is particularly important in light of the fact that sub-optimal solutions may not sufficiently open the eye for successful transfer to decision-directed equalization.

1 Introduction

The objective of this report is to illustrate via simple examples the behavior of the Constant Modulus Algorithm (CMA) used to blindly adapt a fractionally-spaced equalizer (FSE). This effort is motivated by the need for a comprehensive theory of CMA convergence, as found for the popular least mean squares algorithm (LMS), due to the recent demand for inexpensive, yet effective, blind equalization techniques (for broadcast and uncooperative communications).

Most of the analysis on CMA found in the literature is based on the following assumptions:

A1. **Length** The equalizer is sufficiently long to perfectly equalize the channel (zero-forcing solution). In the case of a baud-spaced equalizer (BSE), the equalizer must be infinitely long. In the case of a FSE, the equalizer must be at most one $T/2$-spaced tap shorter than the the length of the channel.

A2. **Invertibility** The channel must be *invertible.* In the case of a BSE, there may be no roots on the unit circle, while for a FSE, there may be no common subchannel roots.

A3. **Source distribution** The source is sub-gaussian, with kurtosis less than 3. Gaussian sources, with kurtosis equal to 3, or sources with kurtosis greater than three do not permit equalization of the channel.

A4. **Source correlation** The source is i.i.d.

A5. **Noise** The channel introduces no noise.

These assumptions enable analysis of the algorithm but often restrict the communications system model to impractical settings. Some work, however, has been done towards understanding the behavior of CMA when these conditions are violated. [1] breaks assumptions (A3) and (A4),

[4] deals with assumption (A2), [3] breaks assumption (A1), and [2] attacks the problem in assumption (A5).

This report aims at violating these assumptions as well. In particular, it focuses on the length assumption (A1) and briefly discusses the invertibility (A2) and noise (A5) assumptions.

2 Derivation of the Fractionally-Spaced CMA Cost Function

In this section we will derive an expression for the CMA cost function $J_{CMA}(f_0, \cdots, f_N)$ in equalizer parameter space for a real source of arbitrary kurtosis in the presence of noise. Key variables are denoted as below.

$s(k)$ – T-spaced source (system input)

c_i – $T/2$-spaced channel impulse response coefficient, $i = 0 \cdots M$

$c_i^{(e)}$ – even subchannel channel coefficients: $c_i^{(e)} = c_{2i}$, $i = 0 \cdots M_e$

$c_i^{(o)}$ – odd subchannel channel coefficients: $c_i^{(o)} = c_{2i+1}$, $i = 0 \cdots M_o$

f_i – $T/2$-spaced equalizer impulse response coefficient, $i = 0 \cdots N$

$f_i^{(e)}$ – even subchannel equalizer coefficients: $f_i^{(e)} = f_{2i}$, $i = 0 \cdots N_e$

$f_i^{(o)}$ – odd subchannel equalizer coefficients: $f_i^{(o)} = f_{2i+1}$, $i = 0 \cdots N_o$

h_i – T-spaced combined channel/equalizer impulse response coefficients, $i = 0 \cdots P$

$n(k)$ – channel noise sampled every $T/2$ seconds

$n^{(e)}(k)$ – on-baud noise samples: $n^{(e)}(k) = n(2k)$

$n^{(o)}(k)$ – off-baud noise samples: $n^{(o)}(k) = n(2k+1)$

$y(k)$ – T-spaced system output

$\mathbf{f}$ – equalizer impulse response vector, $\mathbf{f} = (f_0, \cdots, f_N)^t$

$\mathbf{g}$ – channel impulse response vector, $\mathbf{g} = (g_0, \cdots, g_M)^t$

$\mathbf{h}$ – combined channel/equalizer impulse response vector, $\mathbf{h} = (h_0, \cdots, h_P)^t$

The quantities σ_s^2 and σ_n^2 will be used to refer to the source variance and noise variance, respectively. Similarly, $E[s(k)^4]$ and $E[n(k)^4]$ refer to the source and noise fourth moments. Finally, we can define source modulus, source kurtosis, and noise kurtosis (respectively) as

$$\rho = \frac{E[s(k)^4]}{E[s(k)^2]} \tag{1}$$

$$\kappa_s = \frac{E[s(k)^4]}{E[s(k)^2]^2} \tag{2}$$

$$\kappa_n = \frac{E[n(k)^4]}{E[n(k)^2]^2} \tag{3}$$

Note that for a unit-variance M-PAM source with equiprobable symbols, the kurtosis is

$$\kappa_s = \frac{M}{2} \cdot \frac{\sum_{i=0}^{M/2-1}(1+2i)^4}{\left(\sum_{i=0}^{M/2-1}(1+2i)^2\right)^2} \tag{4}$$

Since it can be shown that the combined baud spaced channel/equalizer combination is the sum of two subchannels, each composed of alternating (i.e. even and odd) fractionally-spaced coefficients,

$$h_l = \sum_{i=0}^{N_e} f_{2i} c_{2(l-i)} + \sum_{i=0}^{N_o} f_{2i+1} c_{2(l-1-i)+1} \tag{5}$$

$$= f_l^{(e)} \star c_l^{(e)} + f_{l-1}^{(o)} \star c_{l-1}^{(o)} \tag{6}$$

we can express the system output in terms of the baud-spaced channel/equalizer parameters h_i and the subchannel taps $f_i^{(e)}$ and $f_i^{(o)}$

$$y(k) = \sum_{i=0}^{P} h_i s(k-i) + \sum_{i=0}^{N_e} f_i^{(e)} n^{(e)}(k-i) + \sum_{i=0}^{N_o} f_i^{(o)} n^{(o)}(k-i) \tag{7}$$

Alternatively, we can combine the last two baud-spaced terms in (7) into one fractionally spaced term, which simplifies the subsequent derivation. Thus the output

$$y(k) = \sum_{i=0}^{P} h_i s(k-i) + \sum_{i=0}^{N} f_i n(k-i) \tag{8}$$

The CMA cost function can be expressed in terms of the system output as follows.

$$J_{CMA} = E[(\rho - y^2(k))^2] \tag{9}$$

$$= \rho^2 - 2\rho E[y^2(k)] + E[y^4(k)] \tag{10}$$

$$= \kappa_s^2(\sigma_s^2)^2 - 2\kappa_s\sigma_s^2 E[y^2(k)] + E[y^4(k)] \tag{11}$$

Solving for its components, we can start with $E[y^2(k)]$:

$$E[y^2(k)] = E\left[\left(\sum_{i=0}^{P} h_i s(k-i) + \sum_{i=0}^{N} f_i n(k-i)\right)^2\right] \tag{12}$$

Assuming the signal and noise are mutually independent white processes,

$$E[s(k-i)n(k-j)] = 0 \text{ for all } i, j \tag{13}$$

$$E[s(k-i)s(k-j)] = \sigma_s^2 \text{ for } i = j \tag{14}$$

$$= 0 \text{ for } i \neq j \tag{15}$$

$$E[n(k-i)n(k-j)] = \sigma_n^2 \text{ for } i = j \tag{16}$$

$$= 0 \text{ for } i \neq j \tag{17}$$

and therefore the expectation in (12) reduces to

$$E[y^2(k)] = \sigma_s^2 \sum_{i=0}^{P} h_i^2 + \sigma_n^2 \sum_{i=0}^{N} f_i^2 \tag{18}$$

In expanding $E[y^4(k)]$, it is useful to define the temporary quantities

$$A(k) = \sum_{i=0}^{P} h_i s(k-i) \tag{19}$$

$$B(k) = \sum_{i=0}^{N} f_i n(k-i) \tag{20}$$

where

$$y(k) = A(k) + B(k)$$

Thus we can write the following:

$$\begin{aligned} E[y^4(k)] &= E\left[(A(k)+B(k))^4\right] && (21)\\ &= E\left[A^4(k) + 4A^3(k)B(k)\right.\\ &\quad \left. + 6A^2(k)B^2(k) + 4A(k)B^3(k) + B^4(k)\right] && (22) \end{aligned}$$

Note that the second and forth terms can be discarded (assuming zero-mean and independent noise and source) since

$$E[A^m(k)B^n(k)] = E[A^m(k)]E[B^n(k)] = 0 \text{ for } n \text{ or } m \text{ odd} \tag{23}$$

which leaves

$$E[y^4(k)] = E[A^4(k)] + 6E[A^2(k)]E[B^2(k)] + E[B^4(k)] \tag{24}$$

Using (13) through (17) and the fact that

$$\begin{aligned} E[s(k-i)s(k-j)s(k-l)s(k-m)] &= (\sigma_s^2)^2 \text{ for } i = j \neq l = m && (25)\\ &= (\sigma_s^2)^2 \text{ for } i = l \neq j = m && (26)\\ &= (\sigma_s^2)^2 \text{ for } i = m \neq j = l && (27)\\ &= E[s(k)^4] \text{ for } i = j = l = m && (28) \end{aligned}$$

we obtain the expressions

$$E[A^2(k)] = \sum_{i=0}^{P} \sigma_s^2 h_i^2 \tag{29}$$

$$E[B^2(k)] = \sum_{i=0}^{N} \sigma_n^2 f_i^2 \tag{30}$$

$$E[A^4(k)] = 3\sum_{i=0}^{P} \sum_{j=0,j\neq i}^{P} h_i^2 h_j^2 (\sigma_s^2)^2 + \sum_{i=0}^{P} h_i^4 E[s(k)^4] \tag{31}$$

$$E[B^4(k)] = 3\sum_{i=0}^{N} \sum_{j=0,j\neq i}^{N} f_i^2 f_j^2 (\sigma_n^2)^2 + \sum_{i=0}^{N} f_i^4 E[n(k)^4] \tag{32}$$

It is convenient to express these in terms of the source kurtosis κ_s and a (gaussian) noise kurtosis $\kappa_n = 3$. This results in

$$E[A^4(k)] = (\sigma_s^2)^2 \left[3\sum_{i=0}^{P}\sum_{j=0}^{P} \left[h_i^2 h_j^2 - h_i^4 \right] + \kappa_s \sum_{i=0}^{P} h_i^4 \right] \tag{33}$$

$$\begin{aligned} E[B^4(k)] &= (\sigma_n^2)^2 \left[3\sum_{i=0}^{N}\sum_{j=0}^{N} \left[f_i^2 f_j^2 - f_i^4 \right] + \kappa_n \sum_{i=0}^{N} f_i^4 \right] \\ &= 3(\sigma_n^2)^2 \left[\sum_{i=0}^{N}\sum_{j=0}^{N} f_i^2 f_j^2 \right] \end{aligned} \tag{34}$$

Finally, equations (29) (30) (33) and (34) can be combined to yield

$$\begin{aligned} E[y^4(k)] &= (\sigma_s^2)^2 \left[3\sum_{i=0}^{P}\sum_{j=0}^{P} h_i^2 h_j^2 + (\kappa_s - 3) \sum_{i=0}^{P} h_i^4 \right] + 6\sigma_s^2\sigma_n^2 \sum_{i=0}^{P} h_i^2 \sum_{j=0}^{N} f_j^2 \\ &\quad + 3(\sigma_n^2)^2 \sum_{i=0}^{N}\sum_{j=0}^{N} f_i^2 f_j^2 \end{aligned} \tag{35}$$

and collecting the terms of the CMA cost function gives

$$\begin{aligned} J_{CMA} &= \kappa_s^2(\sigma_s^2)^2 - 2\kappa_s\sigma_s^2 \left[\sigma_s^2 \sum_{i=0}^{P} h_i^2 + \sigma_n^2 \sum_{i=0}^{N} f_i^2 \right] + (\sigma_s^2)^2 \left[3\sum_{i=0}^{P}\sum_{j=0}^{P} h_i^2 h_j^2 + (\kappa_s - 3) \sum_{i=0}^{P} h_i^4 \right] \\ &\quad + 6\sigma_s^2\sigma_n^2 \sum_{i=0}^{P} h_i^2 \sum_{j=0}^{N} f_j^2 + 3(\sigma_n^2)^2 \sum_{i=0}^{N}\sum_{j=0}^{N} f_i^2 f_j^2 \end{aligned} \tag{36}$$

3 Illustrative Examples of Fractionally-Spaced CMA Cost Functions

Using (36), we may now visually study the shape of the CMA surface as a function of a two-tap FSE. Our first example, shown in Fig. 1, uses all five assumptions above. Channel $\mathbf{g}_1 = (0.2, 0.5, 1, -0.1)^t$ has no common subchannel roots, no noise, the source is white with kurtosis $\kappa_s = 1$, and two taps are enough to perfectly equalize the four-tap channel when *off-baud* (odd) output samples are chosen. Choosing off-baud samples requires solving the system

$$\begin{pmatrix} g_1 & g_0 \\ g_3 & g_2 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{37}$$

while choosing *on-baud* (even) samples requires an approximate answer to the system

$$\begin{pmatrix} g_0 & 0 \\ g_2 & g_1 \\ 0 & g_3 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{38}$$

With all assumptions maintained, according to [5] [6], all CMA minima should be global and should perfectly equalize the channel. Looking at the CMA cost function for the channel in Fig.

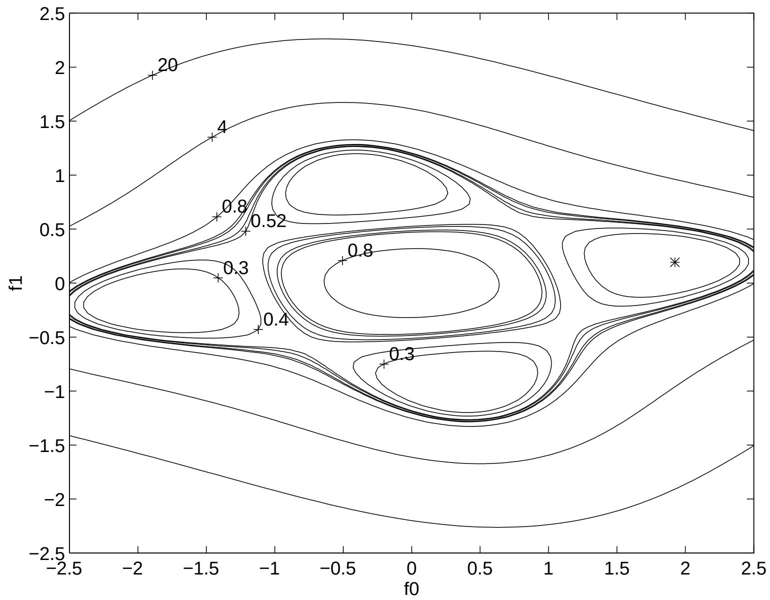

Figure 1: CMA-FSE cost function for channel $\mathbf{g}_1 = (0.2, 0.5, 1, -0.1)^t$. Global minima of $J_{CMA} = 0$ at $\mathbf{f} = \pm(1.9231, 0.1923)^t$ yielding overall channel-FSE response $\mathbf{h} = \pm(1, 0)^t$, and at $\mathbf{f} = \pm(-0.3846, 0.9615)^t$ yielding overall response $\mathbf{h} = \pm(0, 1)^t$. MMSE solution indicated by the "$*$".

1 this is readily verfied. Notice that J_{CMA} has 9 stationary points. At position $(0, 0)$ there is a maximum, and there are four minima for overall channel-FSE combinations $\pm(1, 0)$ and $\pm(0, 1)$. Between each one of these minima is a saddle point, totaling four saddle points in all.

Next, we investigate the assumption concerning common subchannel roots by choosing the channel $\mathbf{g}_2$ which leads to the CMA surface in Fig. 2. This channel, $\mathbf{g}_2 = (0.01, 1, 0.01, -1)^t$, with roots $\{-99.98, -1.0102, 0.9901\}$, has an ill-conditioned convolution matrix requiring an equalizer with *large* coefficients (i.e. high norm). Since the subchannel roots are nearly repeated and not exactly repeated, the channel is still equalizable, and the CMA cost surface resembles a "stretched" version of that in Fig. 1.

4 Examples of Fractionally-Spaced CMA Cost Functions in the Presence of Noise

Next, we observe the effect of noise on the CMA cost function. Using the same channels as the previous examples, we obtain Fig.s 3, 4, and 5. For channel $\mathbf{g}_1$, the effect of the noise is to compress the CMA "bowl", shrinking the convex regions about the minima and increasing the costs at the minima. Since noise gain is proportional to the norm of the equalizer, the minima nearest to the origin (e.g., $\mathbf{f} \approx \pm(0.9326, 0.1365)^t$) sustain the least increase in cost, remaining global minima. Consequently, the other pair of minima become local minima. Note that the equalizer initializations $\mathbf{f} = \pm(1, 0)$ will converge to these false minima.

For channel $\mathbf{g}_2$, with near-common subchannel roots, a similar phenomenon is seen: addition of noise leads to shrinking of the CMA "bowl". Because of the near repetition of roots and the high noise gain inherent in the solutions, a small addition of noise (e.g. 40 dB SNR) results in a

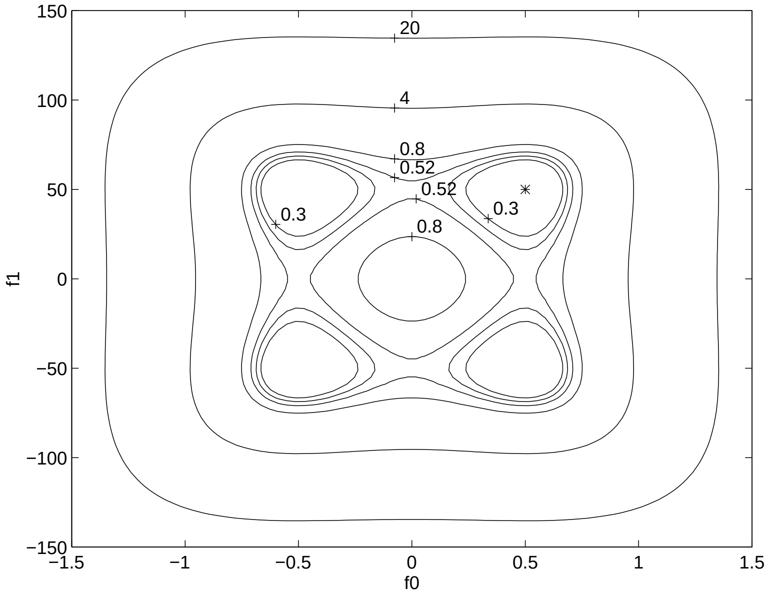

Figure 2: CMA-FSE cost function for channel $\mathbf{g}_2 = (0.01, 1, 0.01, -1)^t$. Global minima of $J_{CMA} = 0$ at $\mathbf{f} = \pm(0.5, 50)^t$ yielding overall channel-FSE response $\mathbf{h} = \pm(1, 0)^t$, and at $\mathbf{f} = \pm(-0.5, 50)^t$ yielding overall response $\mathbf{h} = \pm(0, 1)^t$. Note the extreme scaling of the axes. MMSE solution indicated by the "$*$".

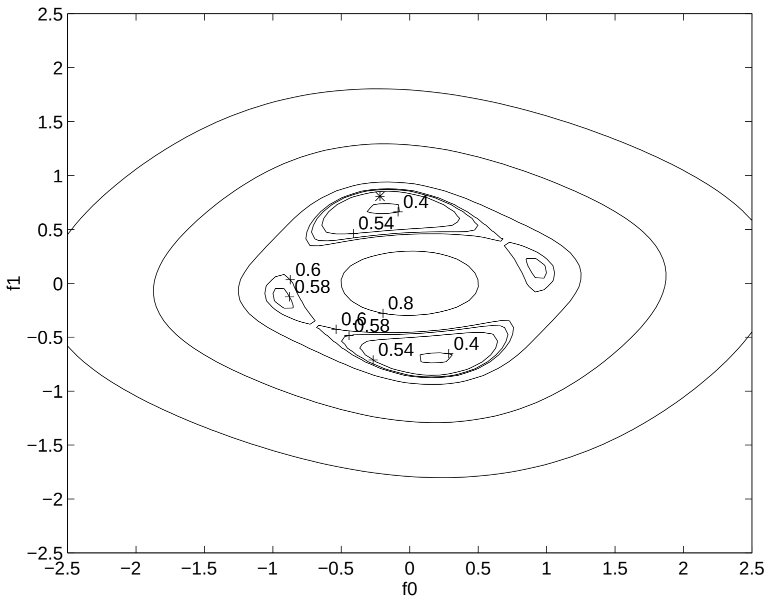

Figure 3: CMA-FSE cost function for channel $\mathbf{g}_1 = (0.2, 0.5, 1.0, -0.1)^t$ with SNR = 7.0 dB. Local minima of $J_{CMA} = 0.5722$ near $\mathbf{f} = \pm(0.9326, 0.1365)^t$ yielding an approximate overall channel-FSE response $\mathbf{h} = \pm(0.4936, 0.0432)^t$, and global minima of $J_{CMA} = 0.3859$ near $\mathbf{f} = (-0.1914, 0.6897)^t$ yielding overall response $\mathbf{h} = \pm(0.0422, 0.7088)^t$. MMSE solution indicated by the "$*$".

significant reduction in the norm of the CMA equalizer solutions. Due to the symmetry in the channel, the discrimination of the two pairs of minima into local and global is not seen.

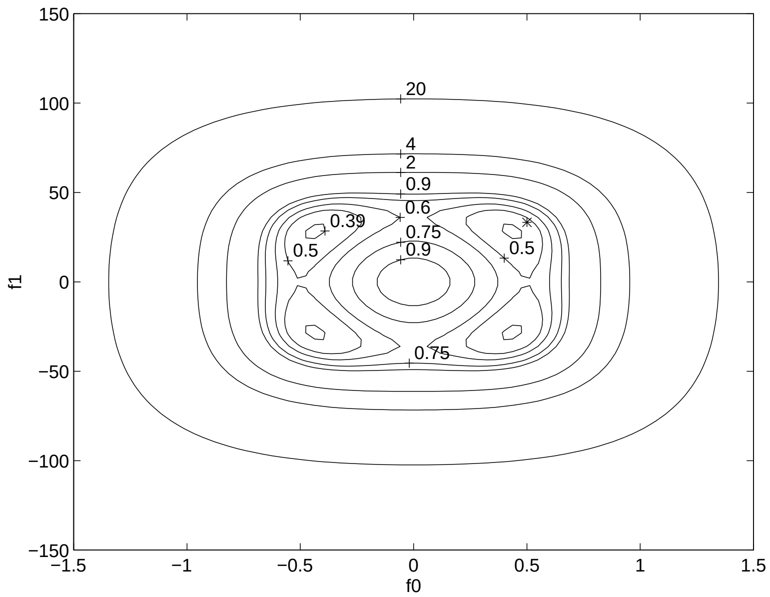

Figure 4: CMA-FSE cost function for channel $\mathbf{g}_1 = (0.01, 1, 0.01, -1)^t$ with SNR = 40 dB. Global minima of $J_{CMA} = 0.3785$ near $\mathbf{f} = \pm(0.4399, 28.1609)^t$ yielding overall channel-FSE response $\mathbf{h} = \pm(0.7215, -0.1582)^t$, and near $\mathbf{f} = \pm(-0.4399, 28.1609)^t$ yielding overall response $\mathbf{h} = \pm(-0.1582, 0.7215)^t$. Note the extreme scaling of the axes. MMSE solution indicated by the "$*$".

Adding a moderate amount of noise to the near-common subchannel-root example (e.g. 7 dB SNR, as in Fig. 3) it is clear that the CMA cost function becomes quite distorted. What was once 4 distinct minima become a single pair yielding *closed-eye* solutions. This is shown in Fig. 5. In effect, the equalizer is compromised more towards noise reduction than towards zero-forcing equalization.

5 Example of an Undermodelled Fractionally-Spaced CMA Cost Function

Finally we turn to the problem of most concern: the case when the FSE is not sufficiently long to perfectly equalize the channel. For clarity, the effects of noise will be omitted in this example. Here are some questions on the behavior of CMA for this case:

- By using an undermodelled FSE, do local minima appear in the cost function?
- For which channel/FSE combination delay does CMA open the eye (if it does open the eye)?
- How far from the minimum mean square error (MMSE) solution are the CMA solutions?
- Which initializations converge to the global minimum?

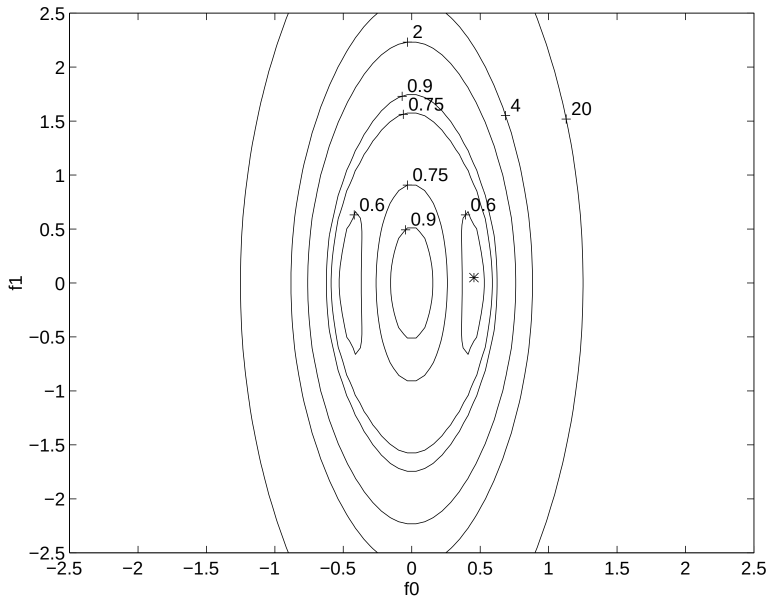

Figure 5: CMA-FSE cost function for channel $\mathbf{g}_1 = (0.01, 1, 0.01, -1)^t$ with SNR = 7 dB. Global minima of $J_{CMA} = 0.45$ near $\mathbf{f} = (\pm 0.45, 0.0)^t$ yielding overall channel-FSE response $\mathbf{h} = \pm(0.45, -0.45)^t$. Note the proportional scaling of the axes. MMSE solution indicated by the "$*$".

The example in Fig. 6 may provide insights into some of these questions. We first confirm the fact that the CMA minima are no longer perfect: there exist a pair of global minima at $\mathbf{f} \approx \pm(-0.7910, 0.7328)^t$, with an FSE setting leading to an open-eye response (for BPSK) $\mathbf{h} = \pm(-0.1640, 0.8119, 0.2082)^t$ and a pair of false minima at $\mathbf{f} = \pm(1.8481, 0.1121)^t$ with an almost closed-eye response $\mathbf{h} = \pm(0.5656, -0.0727, 0.4257)^t$. Note that the equalizer initializations $\mathbf{f} = \pm(1, 0)$ will converge to these false minima.

In order to find a connection with the MMSE design, we calculate for each system delay δ the MMSE equalizer $\mathbf{f}_\delta$:

$$\mathbf{f}_0 = \begin{pmatrix} 2.1368 \\ 0.0285 \end{pmatrix}, \mathbf{f}_1 = \begin{pmatrix} -0.8889 \\ 0.8148 \end{pmatrix}, \mathbf{f}_2 = \begin{pmatrix} 1.3504 \\ 0.3647 \end{pmatrix}, \tag{39}$$

along with the corresponding channel/FSE combination $\mathbf{h}_\delta$:

$$\mathbf{h}_0 = \begin{pmatrix} 0.6439 \\ -0.1852 \\ 0.4416 \end{pmatrix}, \mathbf{h}_1 = \begin{pmatrix} -0.1852 \\ 0.9037 \\ 0.2296 \end{pmatrix}, \mathbf{h}_2 = \begin{pmatrix} 0.4416 \\ 0.2296 \\ 0.4524 \end{pmatrix}, \tag{40}$$

and the resulting mean square error, MSE_δ,

$$\text{MSE}_0 = 0.3561,\ \text{MSE}_1 = 0.0963,\ \text{MSE}_2 = 0.5476. \tag{41}$$

From these calculations and the cost function shown in Fig. 6, one could hypothesize that in the case of an undermodelled FSE, the global CMA minima are located near the MMSE minima which correspond to the optimum delays. Thus, the global minima are close to $\mathbf{f}_1$ and the local minima are closer to $\mathbf{f}_0$ and $\mathbf{f}_2$. However, there still remains the question of whether spurious local minima (i.e. not clearly associated with any MMSE solution) ever appear when the FSE is undermodelled.

If we can establish that all CMA minima are *close* to a particular MMSE minimum, then it may be possible to devise a way to initialize CMA so as to converge to the global minimum – the minimum supposedly associated with the MMSE solution optimized over all possible system delays.

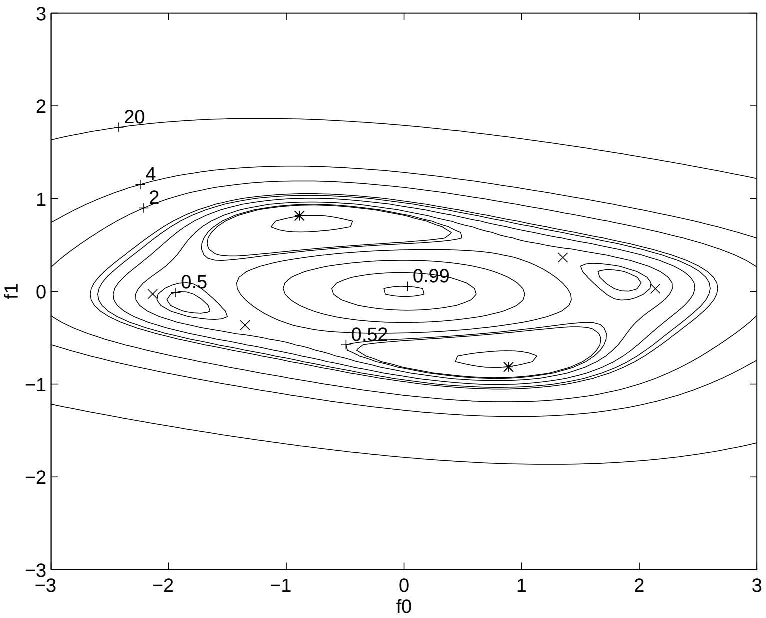

Figure 6: CMA-FSE cost function for channel $\mathbf{g} = (0.1, 0.3, 1.0, -0.1, -0.5, 0.2)^t$. Global minima of $J_{CMA} = 0.4861$ near $\mathbf{f} = \pm(1.8481, 0.1121)^t$ yielding overall channel-FSE response $\mathbf{h} = \pm(0.5656, -0.0727, 0.4257)^t$, and local minima of $J_{CMA} = 0.2631$ near $\mathbf{f} = \pm(-0.7910, 0.7328)^t$ yielding overall response $\mathbf{h} = \pm(-0.1640, 0.8119, 0.2082)^t$. Note the the $*$'s locate the MMSE solutions corresponding to optimum system delays, and that the $\times$'s locate the MMSE solutions for other delays.

6 Conclusion

Various examples have shown the existance of local minima. Should equalizer initialization be chosen within the region of attraction of any of these local minima, the CMA-FSE algorithm will ill-converge. This is potentially disastrous when such false minima correspond to closed-eye solutions. This emphasizes the need for techniques that avoid false minima or, conversely, converge to the global minimum.

7 Appendix: MATLAB Code

```
% CMA_err_surf.m
%
% This program plots the MSE and CMA cost contours of 2-tap BSEs/FSEs
```

```
% for a specified channel, in the presence of noise, and for a white
% source of specified kurtosis.  In addition, it plots the major
% and minor axes of the contours of the MSE surface, as well as the
% Wiener equalizers for various system delays.  The solution
% corresponding to the optimal delay (or one of them, should there
% be more than one) is indicated by an asterisk.
%
% Phil Schniter
% Blind Equalization Research Group
% Cornell University
% last modified 9/30/96

 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
 % user defined source parameters %
 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

 sigma2_s = 1; % signal power
 kappa = 1.0; % source kurtosis

 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
 % user defined channels and related parameters %
 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
 clear v;

 % THESE PARAMETERS MUST BE SUPPLIED FOR EACH SIMULATION:
 % -----------------------------------------------------
 % useFSE   : {0,1} corresponds to {BSE,FSE}
 % B        : channel impulse response
 % sigma2_n : noise variance
 % range_f1 : +/- span of contour plot along 1st equalizer parameter
 % range_f2 : +/- span of contour plot along 2nd equalizer parameter
 % v        : (optional : user-defined levels of contour plot)
 %
 % SIMPLY UN-COMMENT ONE OF THE EXAMPLES BELOW, OR ADD YOUR OWN...

  % BSE channel
  %useFSE=0;
  %B = [0.5,-0.3,0.25];
  %v=[0.3 0.4 0.52 0.54 0.58 0.6 0.8 4 20]; %BSE
  %sigma2_n = 0.0; % FSE1
  %range_f1 = 2.0; % grid span along f1
  %range_f2 = 2.0; % grid span along f2
  %[0.0600   -1.0733];   0.5188;
  %[-1.6397   -0.9326];  0.2628;

  % FSE1: Perfectly Equalizable
  %useFSE=1;
  %B = [0.2,0.5,1,-0.1]; % FSE1 & FSE3
```

```
%v=[0.3 0.4 0.52 0.54 0.58 0.6 0.8 4 20]; %FSE1
%sigma2_n = 0.0; % FSE1
%range_f1 = 2.5; % grid span along f1
%range_f2 = 2.5; % grid span along f2

% FSE2: Nearly Reflected Roots
%useFSE=1;
%B = [0.01 1 0.01 -1.0]';
%v=[0.3 0.4 0.52 0.8 4 20]; %FSE2
%sigma2_n = 0.0; % FSE1
%range_f1 = 1.5; % grid span along f1
%range_f2 = 150.0; % grid span along f2

% FSE3: Perfectly Equalizable with noise
%useFSE=1;
%B = [0.2, 0.5, 1.0, -0.1]; % FSE1 & FSE3
%v = [0.4 0.54, 0.58, 0.6, 0.8 4,20]; % FSE3
%sigma2_n = 0.2; % FSE3
%range_f1 = 2.5; % grid span along f1
%range_f2 = 2.5; % grid span along f2
%[1.3285;  0.4343]; 0.5073;
%[-1.0990; 0.7969];  0.1341;

% FSE4: Nearly Reflected Roots with a little noise
%useFSE=1;
%B = [0.01 1 0.01 -1.0]';
%v = [0.39 0.5 0.6 0.75 0.9 2 4 20];
%sigma2_n = 0.0001; % FSE4
%range_f1 = 1.5; % grid span along f1
%range_f2 = 150.0; % grid span along f2
%[0.4964; 7.9617]; 0.4872;
%[0.4964; -7.9617]; 0.4872;

% FSE4b: Nearly Reflected Roots with more noise
%useFSE=1;
%B = [0.01 1 0.01 -1.0]';
%v = [0.39 0.5 0.6 0.75 0.9 2 4 20];
%sigma2_n = 0.2; % FSE4b
%range_f1 = 2.5; % grid span along f1
%range_f2 = 2.5; % grid span along f2
%[-0.4501, 0.0063]; 0.5404;

% FSE5: Undermodelled Channel with no noise
B = [0.1,0.3,1,-0.1 0.5 0.2]'; % FSE1 & FSE3
%v = [0.5 0.52 0.6 0.75 0.9 0.99 2 4 20];
sigma2_n = 0;
range_f1 = 3.0; % grid span along f1
range_f2 = 3.0; % grid span along f2
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% NO user parameters below this point! %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% channel and equalizer lengths
N_g = length(B); % BS/FS channel length
N_f = 2; % BS/FS equalizer length

% build channel
B=B(:);
GG = convmtx(B,N_f); % channel convolution matrix
if useFSE,
  G = GG(2:2:N_g+N_f-1,:); % create decimated FS chan conv matrix
else
  G = GG; % keep BS chan conv matrix
end

% solve for optimal delay MSE equalizer
N_h = length(G(:,1));
lambda = sigma2_n/sigma2_s;
P = inv(G'*G+lambda*eye(N_f));
F = (eye(N_h) - G*P*G')^2 + lambda*G*P'*P*G';
[mse_opt,delta_opt] = min(diag(F));
h_opt = zeros(N_h,1); h_opt(delta_opt)=1;
PGt = P*G';
f_opt = PGt*h_opt; % MSE optimal equalizer

% define a grid in f space
if length(f_opt) ~= 2, error('plots only for equalizer lengths of 2'); end;
grid_density = 5; % grid density
num_pts = 80;
f1_grid = linspace(-range_f1,range_f1,num_pts);
f2_grid = linspace(-range_f2,range_f2,num_pts);
L1 = length(f1_grid); L2 = length(f2_grid);

% calculate CMA cost values on grid
CMA = zeros(L1,L2);
for i=1:L1,
  for k=1:L2,
    f = [f1_grid(i);f2_grid(k)];
    h = G*f;
    hTh = h'*h;  fTf = f'*f;
    CMA(i,k) = kappa^2*sigma2_s^2 - 2*kappa*sigma2_s^2*hTh...
- 2*kappa*sigma2_s*sigma2_n*fTf...
+ 3*sigma2_s^2*hTh^2 + sigma2_s^2*(kappa-3)*(h.^2)'*(h.^2)...
+ 6*sigma2_s*sigma2_n*hTh*fTf + 3*sigma2_n^2*fTf^2;
  end
```

```
end

% plot CMA contours
figure(1);
if exist('v')~=1,
  %v = 1.1*min(min(CMA))+log10([[1:0.15:1.8],[2:0.4:7.6],[8:1:11],[12:2:20]]);
  v = min(min(CMA))+log10([[1:0.15:1.8],[2:0.4:20]]);
end;
cs = contour(f1_grid,f2_grid,CMA',v,'c');
hold on;
for delta = 1:N_h,  % plot optimal equalizers for delta delay
  plot(PGt(1,delta),PGt(2,delta),'x');
  plot(-PGt(1,delta),-PGt(2,delta),'x');
end;
plot(PGt(1,delta_opt),PGt(2,delta_opt),'*');
plot(-PGt(1,delta_opt),-PGt(2,delta_opt),'*');
hold off;
xlabel('f0'); ylabel('f1');
title('CMA Error Surface');

% plot MSE ellipse axes
hold on;
origin = [-(3/4)*range_f1,(3/4)*range_f2];
[V,lambda] = eig(G'*G+lambda*eye(N_f))
vec1 = V(:,1)*sqrt(1./lambda(1,1))/4;
vec2 = V(:,2)*sqrt(1./lambda(2,2))/4;
plot(origin(1)+[-vec1(1),vec1(1)],origin(2)+[-vec1(2),vec1(2)]);
plot(origin(1)+[-vec2(1),vec2(1)],origin(2)+[-vec2(2),vec2(2)]);
text((1.21)*origin(1),(1.20)*origin(2),'MSE ellipse axes');
hold off;
```

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