HOMEWORK SOLUTIONS #4

1. (a) Here the 2×2 autocorrelation matrix for $\{u(n))\}$ equals $\boldsymbol{R} = \boldsymbol{H}^H \boldsymbol{H} + 0.01\boldsymbol{I} = \left[\begin{smallmatrix} 1.01 & 0.5 \\ 0.5 & 1.01 \end{smallmatrix}\right]$ since $\boldsymbol{H} = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{smallmatrix}\right]^t$. The eigenvalues of $\boldsymbol{R}$ equal $\{1.51, 0.51\}$, so that the maximum stepsize is $\mu_{\max} = \frac{2}{\lambda_{\max}} = 1.3245$.

   (b) Matlab code returns the GD-MSE parameter trajectories in Fig. 1 and the learning curves in Fig. 2.

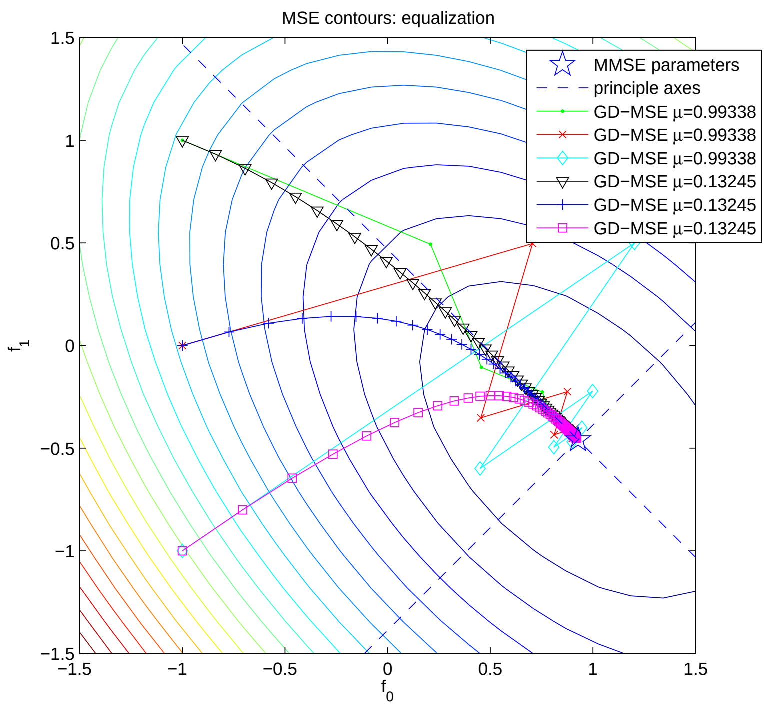

Figure 1: GD-MSE equalization trajectories

   (c) Note that for $\mu = \frac{1}{10}\mu_{\max}$, the trajectories smoothly follow the gradient path (perpendicular to the cost contours). The trajectories converge quickly in the direction of the eigenvector corresponding to the large eigenvalue and slowly in the direction of the eigenvector corresponding to the small eigenvalue. For $\mu = \frac{3}{4}\mu_{\max}$, the trajectories don't follow the gradient: they jump around quite a bit. This is quite a large stepsize, but still within the bounds that guarantee GD-MSE convergence.

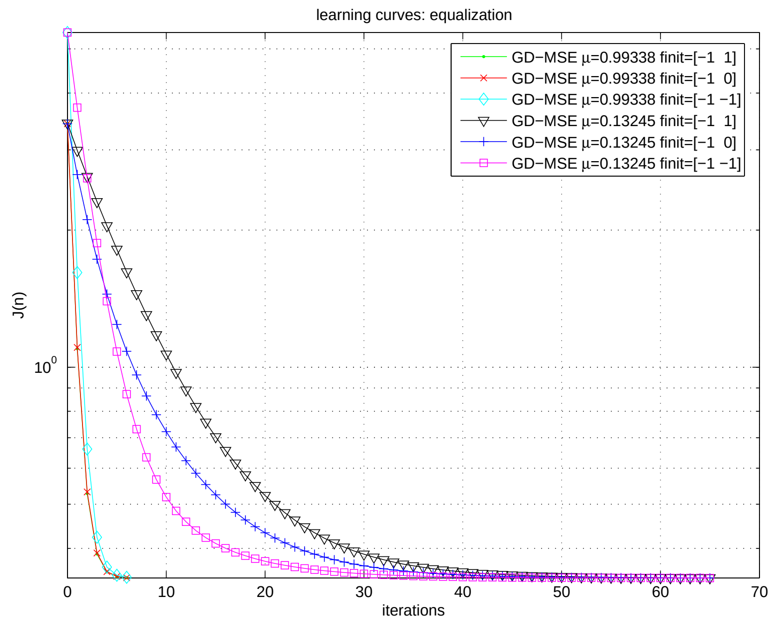

Figure 2: GD-MSE equalization learning curves

2. (a) Defining $\boldsymbol{s}(n) = [s(n), s(n-1), \ldots, s(n-K+1)]^t$, we see that

$$\begin{aligned} e(n) &= \big(\boldsymbol{f} - \boldsymbol{h}\big)^H \boldsymbol{s}(n) - w(n) \\ \mathrm{E}\{|e(n)|^2\} &= \big(\boldsymbol{f} - \boldsymbol{h}\big)^H \mathrm{E}\{\boldsymbol{s}(n)\boldsymbol{s}^H(n)\}\big(\boldsymbol{f} - \boldsymbol{h}\big) - \big(\boldsymbol{f} - \boldsymbol{h}\big)^H \mathrm{E}\{\boldsymbol{s}(n)w^*(n)\} \\ &\quad - \mathrm{E}\{w(n)\boldsymbol{s}^H(n)\}\big(\boldsymbol{f} - \boldsymbol{h}\big) + \sigma_w^2 \\ &= \big(\boldsymbol{f} - \boldsymbol{h}\big)^H \boldsymbol{R}_s \big(\boldsymbol{f} - \boldsymbol{h}\big) + \sigma_w^2 \end{aligned}$$

The MMSE solution is

$$\boldsymbol{f}_\star = \boldsymbol{h}$$

regardless of $\boldsymbol{R}_s$ and σ_w^2. When $\boldsymbol{R}_s$ is singular, however, $\boldsymbol{f}_\star$ is not unique. Since input autocorrelation matrix equals $\boldsymbol{R}_s = \boldsymbol{I}$, we have both eigenvalues equal to 1 and maximum stepsize $\mu_{\max} = \frac{2}{\lambda_{\max}} = 2$.

(b) Matlab code returns the GD-MSE parameter trajectories in Fig. 3 and the learning curves in Fig. 4.

(c) From *any* initializaiton, the gradient descent trajectories flow directly towards the Wiener solution since the cost contours are perfectly round. This is a consequence of the fact that the filter input process is white (i.e., its autocorrelation matrix has equal eigenvalues). From a convergence-speed perspective, this is a more advantageous situation than that of the equalization application. There, $H^*(z)$ colored the equalizer input, yielding an autocorrelation matrix eigenvalue disparity and hence a convergence slow-down in the eigenvector-direction corresponding to the smallest eigenvalue.

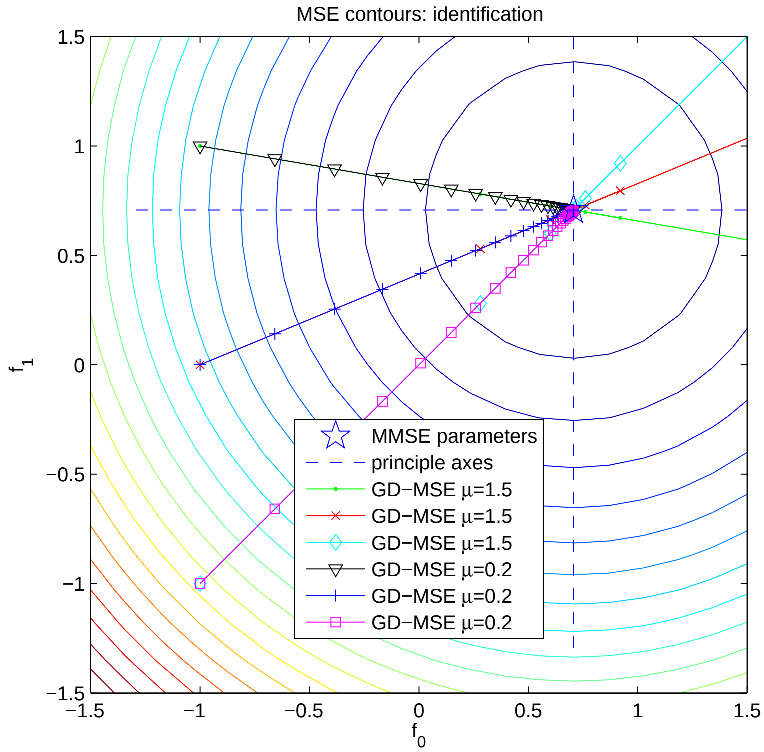

Figure 3: GD-MSE identification trajectories

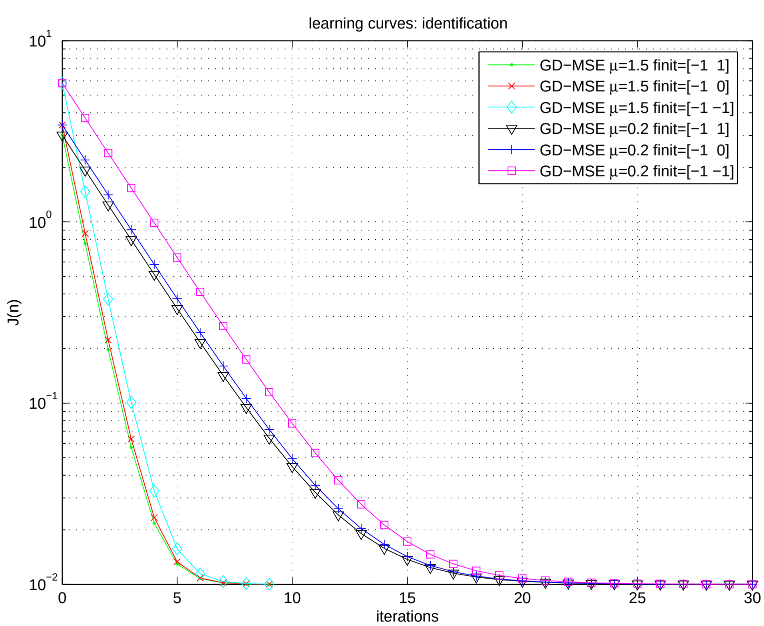

Figure 4: GD-MSE identification learning curves