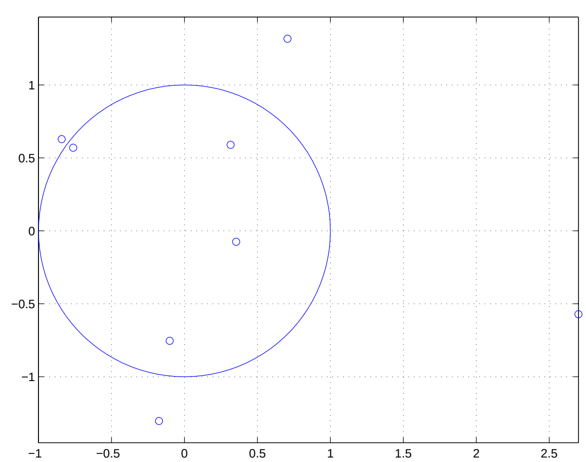
HOMEWORK SOLUTIONS #3

1. (a) Using $\sigma_v^2 = 0.5$ and the randomly generated coefficients below for $\boldsymbol{b}$, we obtain the autocorrelation sequence $\boldsymbol{r}$ shown below. Note that it is conjugate symmetric (i.e., $r(k) = r^*(-k)$).

```
b =
   1.0000
  -0.1961 + 1.0579i
   0.1681 + 0.3363i
   0.3307 + 0.7559i
  -0.0373 - 0.3903i

r =
  -0.0112 - 0.1171i
  -0.0225 + 0.2616i
   0.2296 - 0.0644i
   0.0388 + 0.2187i
   0.9401
   0.0388 - 0.2187i
   0.2296 + 0.0644i
  -0.0225 - 0.2616i
  -0.0112 + 0.1171i
```

   (b) The roots of $B*(z)$ are plotted in complex plane below. Note the symmetry across the unit circle.



   (c) Building polynomial $C^*(z)$ from the minimum phase roots and choosing the corresponding σ_w^2 yields the coefficients below. It was verified that these coefficients generate exactly the autocorrelation $\boldsymbol{r}$ above.

```
c =
   1.0000
   0.1933 + 0.3312i
   0.1783 - 0.1347i
   0.0276 + 0.4190i
  -0.0167 - 0.1749i

sig2w =
   0.6695
```

(d) Note that $\boldsymbol{c} \neq \boldsymbol{b}$ and that $\sigma_w^2 \neq 0.5$ though both MA models generate the same autocorrelation sequence. Thus, while these models are not equal, we could say that they are equivalent.

We can understand the situation in this way: Every finite-lag autocorrelation sequence $\{r(k)\}$ leads to a set of zeros with unit-circle symmetry. An MA model constructed using *any* one zero from each pair (and appropriate noise variance) will produce a random process with autocorrelation $\{r(k)\}$. Since there are many ways to choose one zero from each pair, there will be many MA models yielding the same autocorrelation. In other words, the choice of minimum-phase roots was not necessary to solve the problem; we could have chosen "maximum phase" roots (i.e., those outside the unit circle) or some "mixed phase" combination.

2. (a) The MSE cost is

$$\begin{aligned}
e(n) &= y(n) - s(n-\Delta) \\
&= (\boldsymbol{H}\boldsymbol{f} - \boldsymbol{e}_\Delta)^H \boldsymbol{s}(n) - \boldsymbol{f}^H \boldsymbol{w}(n) \\
J_{\text{mse}} &= \mathrm{E}\{|e(n)|^2\} \\
&= \mathrm{E}\{(\boldsymbol{H}\boldsymbol{f} - \boldsymbol{e}_\Delta)^H \boldsymbol{s}(n)\boldsymbol{s}^H(n)(\boldsymbol{H}\boldsymbol{f} - \boldsymbol{e}_\Delta)\} + \mathrm{E}\{\boldsymbol{f}^H \boldsymbol{w}(n)\boldsymbol{w}^H(n)\boldsymbol{f}\} \\
&\quad \text{since } \{s(n)\} \text{ and } \{w(n)\} \text{ are uncorrelated.} \\
&= (\boldsymbol{H}\boldsymbol{f} - \boldsymbol{e}_\Delta)^H \mathrm{E}\{\boldsymbol{s}(n)\boldsymbol{s}^H(n)\}(\boldsymbol{H}\boldsymbol{f} - \boldsymbol{e}_\Delta) + \boldsymbol{f}^H \mathrm{E}\{\boldsymbol{w}(n)\boldsymbol{w}^H(n)\}\boldsymbol{f} \\
&= (\boldsymbol{H}\boldsymbol{f} - \boldsymbol{e}_\Delta)^H \boldsymbol{R}_s(\boldsymbol{H}\boldsymbol{f} - \boldsymbol{e}_\Delta) + \boldsymbol{f}^H \boldsymbol{R}_w \boldsymbol{f}
\end{aligned}$$

(b) At the point $\boldsymbol{f} = \boldsymbol{f}_\star$, we know $\nabla_{\boldsymbol{f}} J_{\text{mse}} = \boldsymbol{0}$. So, setting

$$\nabla_{\boldsymbol{f}} J_{\text{mse}} = 2\boldsymbol{H}^H \boldsymbol{R}_s(\boldsymbol{H}\boldsymbol{f} - \boldsymbol{e}_\Delta) + 2\boldsymbol{R}_w \boldsymbol{f}$$

to zero implies

$$\boldsymbol{f}_\star = (\boldsymbol{H}^H \boldsymbol{R}_s \boldsymbol{H} + \boldsymbol{R}_w)^{-1} \boldsymbol{H}^H \boldsymbol{R}_s \boldsymbol{e}_\Delta.$$

Notice $\boldsymbol{H}^H \boldsymbol{R}_s \boldsymbol{H} + \boldsymbol{R}_w = \mathrm{E}\{\boldsymbol{u}(n)\boldsymbol{u}^H(n)\} := \boldsymbol{R}_u$. We can solve for the MMSE by plugging $\boldsymbol{f}_\star$ into J_{mse}:

$$\begin{aligned}
J_{\text{mse}} &= \boldsymbol{e}_\Delta^H(\boldsymbol{R}_s^H \boldsymbol{H}\boldsymbol{R}_u^{-1}\boldsymbol{H}^H - \boldsymbol{I})\boldsymbol{R}_s(\boldsymbol{H}\boldsymbol{R}_u^{-1}\boldsymbol{H}^H\boldsymbol{R}_s - \boldsymbol{I})\boldsymbol{e}_\Delta + \boldsymbol{e}_\Delta^H \boldsymbol{R}_s^H \boldsymbol{H}\boldsymbol{R}_u^{-1}\boldsymbol{R}_w\boldsymbol{R}_u^{-1}\boldsymbol{H}^H\boldsymbol{R}_s\boldsymbol{e}_\Delta \\
&= \boldsymbol{e}_\Delta^H \boldsymbol{R}_s^H \boldsymbol{H}\boldsymbol{R}_u^{-1}\boldsymbol{R}_u\boldsymbol{R}_u^{-1}\boldsymbol{H}^H\boldsymbol{R}_s\boldsymbol{e}_\Delta - 2\boldsymbol{e}_\Delta^H\boldsymbol{R}_s^H\boldsymbol{H}\boldsymbol{R}_u^{-1}\boldsymbol{H}^H\boldsymbol{R}_s\boldsymbol{e}_\Delta + \boldsymbol{e}_\Delta^H\boldsymbol{R}_s\boldsymbol{e}_\Delta \\
&= \boldsymbol{e}_\Delta^H\boldsymbol{R}_s\boldsymbol{e}_\Delta - \boldsymbol{e}_\Delta^H\boldsymbol{R}_s^H\boldsymbol{H}\boldsymbol{R}_u^{-1}\boldsymbol{H}^H\boldsymbol{R}_s\boldsymbol{e}_\Delta \\
&= \sigma_s^2 - \boldsymbol{e}_\Delta^H\boldsymbol{R}_s^H\boldsymbol{H}(\boldsymbol{H}^H\boldsymbol{R}_s\boldsymbol{H} + \boldsymbol{R}_w)^{-1}\boldsymbol{H}^H\boldsymbol{R}_s\boldsymbol{e}_\Delta.
\end{aligned}$$

(c) Principle axes have directions $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ and lengths $\{\sqrt{\lambda_1^{-1}}, \sqrt{\lambda_2^{-1}}\}$, where $\boldsymbol{R}_u = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^H$ denotes the eigendecomposition of $\boldsymbol{R}_u$ and where $\boldsymbol{U} = [\boldsymbol{u}_1 \boldsymbol{u}_2]$ and $\boldsymbol{\Lambda} = [\begin{smallmatrix}\lambda_1 & \\ & \lambda_2\end{smallmatrix}]$. MSE cost contours are plotted below.

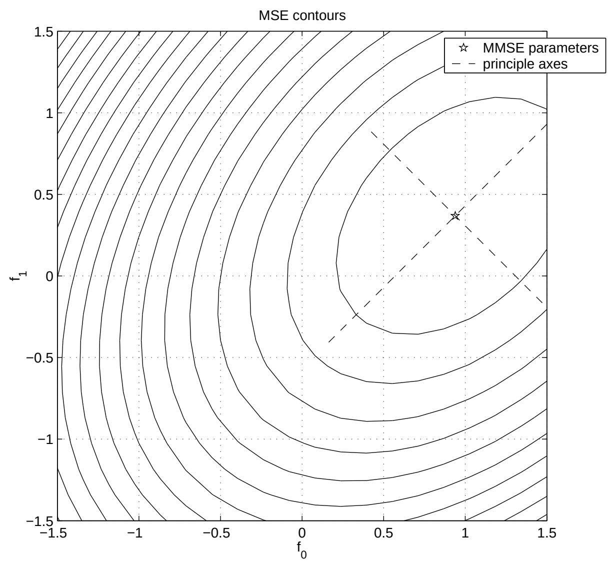


Figure 1: MSE cost contours for $\Delta = 0$.

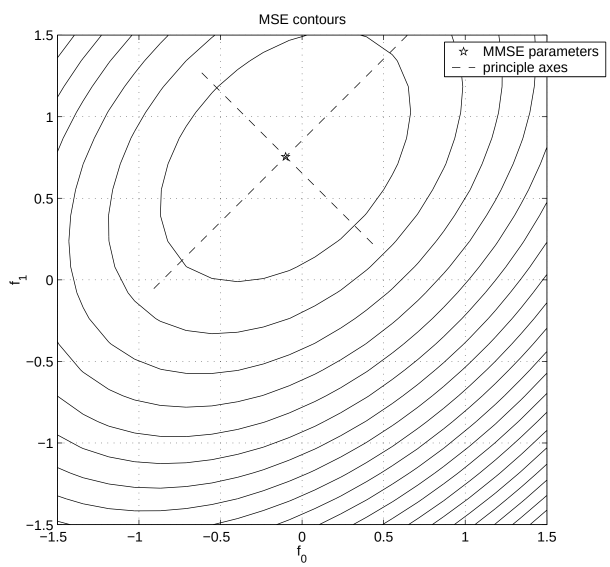


Figure 2: MSE cost contours for $\Delta = 1$.

(d) The AR-1 noise model is specified by the feedback gain $a_1 = -r_w(0)^{-1} r^*(1)$ and the driving noise variance $\sigma_v^2 = \sum_{i=0}^{1} a_i r(i)$ (where $a_0 = 1$).

Note: the experimental $\widehat{J_{\min}}$ values below are based on psuedorandom numbers and thus change from simulation to simulation.

Δ	$J_{\min}$	$\widehat{J_{\min}}$
0	0.0615	0.0617
1	0.1956	0.1956
2	0.7654	0.7632

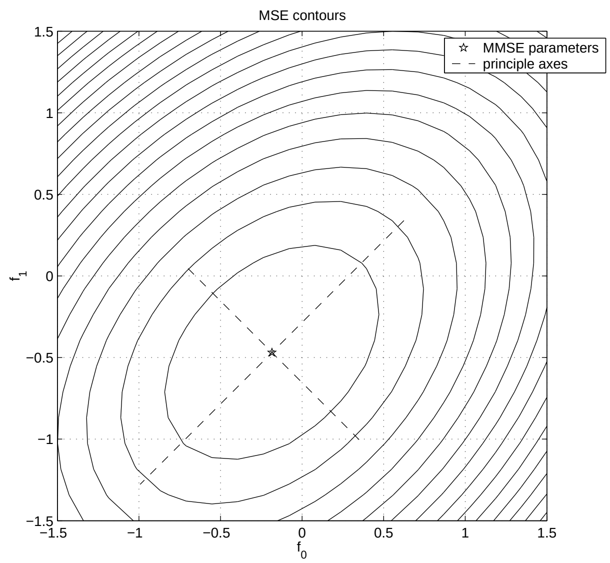


Figure 3: MSE cost contours for $\Delta = 2$.

3. From $J_{\rm mfe} = \mathrm{E}\{|y(n) - d(n)|^4\} = \mathrm{E}\{(|y(n) - d(n)|^2)^2\}$ we have

$$\begin{aligned}
\nabla_{\boldsymbol{f}} J_{\rm mfe} &= \mathrm{E}\{2|y(n) - d(n)|^2 \cdot \nabla_{\boldsymbol{f}} |y(n) - d(n)|^2\} \\
&= \mathrm{E}\{2|y(n) - d(n)|^2 [\nabla_{\boldsymbol{f}} (y(n) - d(n)) \cdot (y(n) - d(n))^* + \nabla_{\boldsymbol{f}} (y(n) - d(n))^* \cdot (y(n) - d(n))]\}
\end{aligned}$$

where

$$\begin{aligned}
\nabla_{\boldsymbol{f}} (y(n) - d(n)) &= \nabla_{\boldsymbol{f}} (\boldsymbol{f}^H \boldsymbol{u}(n) - d(n)) = 2\boldsymbol{u}(n) \\
\nabla_{\boldsymbol{f}} (y(n) - d(n))^* &= \nabla_{\boldsymbol{f}} (\boldsymbol{u}^H(n)\boldsymbol{f} - d^*(n)) = \mathbf{0}
\end{aligned}$$

so that

$$\nabla_{\boldsymbol{f}} J_{\rm mfe} = 4\,\mathrm{E}\{\boldsymbol{u}(n)|y(n) - d(n)|^2 (y(n) - d(n))^*\}.$$

With the instantaneous gradient approximation, we arrive at the LMF algorithm:

$$\begin{aligned}
\boldsymbol{f}(n+1) &= \boldsymbol{f}(n) - \frac{\mu}{2} \nabla_{\boldsymbol{f}} J_{\rm mfe} \\
&= \boldsymbol{f}(n) - 2\mu \boldsymbol{u}(n)|y(n) - d(n)|^2 (y(n) - d(n))^*
\end{aligned}$$