HOMEWORK ASSIGNMENT #3

Due Wed. Oct. 12, 2005 (in class)

1. As derived in class, the autocorrelation sequence of a MA process generated by the length-K FIR system $B^*(z)$ and driven by σ_v^2-variance white noise can be computed from

$$r(l) = \sum_{i=0}^{K-1-l} b_{i+l}^* b_i \sigma_v^2. \qquad (1)$$

If, on the other hand, we are given $\{r(l)\}$ and asked to solve for $\{b_i\}$ and σ_v^2, we have no simple equation. Instead, we use *spectral factorization* to solve the problem. The basic procedure is

- Compute the roots $\{\rho_i\}_{i=1}^{2(K-1)}$ of the polynomial $R(z) = \sum_{i=-K+1}^{K-1} r(i) z^i$. A valid autocorrelation sequence will always have roots that come in pairs mirrored across the unit circle in the complex plane.
- Choose the subset of roots $\{\rho_i{}'\}_{i=1}^{K-1}$ which lie inside the unit circle. These are called *minimum-phase* roots. (If $R(z)$ has any root pairs *on* the unit circle, put one from each pair in $\{\rho_i'\}$.)
- From the minimum phase roots, construct $B^*(z) = \prod_{i=1}^{K-1}(1 - z/\rho_i') = \sum_{i=0}^{K-1} b_i^* z^i$. Note that $\{\rho_i'\}$ are the roots of $B^*(z)$ and that $b_0 = 1$.
- Set $\sigma_v^2 = r(0)\left(\sum_{i=0}^{K-1} |b_i|^2\right)^{-1}$.

For this problem, do the following:

(a) Create a MA model characterized by $\{\sigma_v^2, b_1, b_2, \ldots, b_K\}$, where $\sigma_v^2 = 0.5$ and $\{b_i\}_{i=1}^4$ are randomly chosen complex numbers. (As always, $b_0 = 1$.) Compute the corresponding autocorrelation sequence $\{r(l)\}$ using (1).

(b) Using the minimum-phase spectral factorization procedure outlined above, compute another MA model $\{\sigma_w^2, c_1, c_2, \ldots, c_K\}$ to match $\{r(l)\}$.

(c) Using (1), compute the autocorrelation sequence of the MA model $\{\sigma_w^2, c_1, c_2, \ldots, c_K\}$.

(d) Does $\{\sigma_v^2, b_1, b_2, \ldots, b_K\} = \{\sigma_w^2, c_1, c_2, \ldots, c_K\}$? Should they be equal? Comment.

Useful MATLAB commands are: `randn, roots, poly, find`.

2. Communication channels are often modelled as LTI FIR systems plus an additive noise (see Fig. 1). Here we investigate the design of MMSE equalizers.

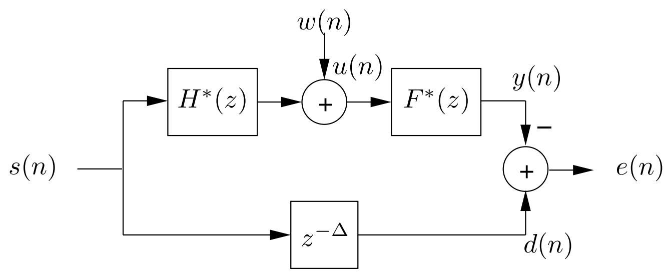

Figure 1: Linear Channel Equalization

(a) Derive an expression for the MSE cost $J_{\text{mse}} = \text{E}\{|e(n)|^2\}$ in terms of the equalizer coefficient vector $\boldsymbol{f} = [f_0, f_1, \ldots, f_M]^t$, the source and noise autocorrelation matrices ($\boldsymbol{R}_s$ and $\boldsymbol{R}_w$), the delay Δ, and the channel convolution matrix $\boldsymbol{H}$ composed from the impulse response $\boldsymbol{h} = [h_0, h_1, \ldots, h_L]^t$. Assume that $\{w(n)\}$ is uncorrelated with $\{s(n)\}$.

(b) Using the MSE cost derived above, find an expression for the MMSE equalizer vector ($\boldsymbol{f}_\star$) and the MMSE ($J_{\min}$). Assume that $(\boldsymbol{H}^H \boldsymbol{R}_s \boldsymbol{H} + \boldsymbol{R}_w)^{-1}$ exists.

(c) Assuming a two-tap equalizer, the two-tap channel

$$\boldsymbol{h} = [h_0, h_1]^t = [1, -0.5]^t,$$

and autocorrelation matrices

$$\boldsymbol{R}_w = \begin{pmatrix} 0.01 & 0.005 \\ 0.005 & 0.01 \end{pmatrix}, \qquad \boldsymbol{R}_s = \boldsymbol{I},$$

calculate $J_{\min}$ and plot the MSE contours as a function of the two equalizer parameters for the choices $\Delta = 0, 1, 2$ and f_k between -1.5 and 1.5. Superimpose the principle ellipse axes and mark the location of the optimal parameters. The result should look something like Fig. 2. Make sure to use equally scaled x and y axes! (Hint: Derive an expression for the equalizer input autocorrelation matrix and compute its eigenvalues/vectors. Use `eig, contour, plot, hold, axis equal`.)

(d) For the same parameters as part (c), experimentally estimate the MMSE costs by generating sequences $\{w(n)\}$ and $\{s(n)\}$ (with specified autocorrelation properties), calculating the MMSE equalizer $\boldsymbol{f}_\star$, and using them to compute $\{e(n)\}$ of length $N = 10^5$. Then compute $\widehat{J_{\min}} = \frac{1}{N}\sum_{n=1}^{N} |e(n)|^2$. (Hint: $\{w(n)\}$ may be generated by an AR-1 model whose parameters are determined by the Yule-Walker equations.)

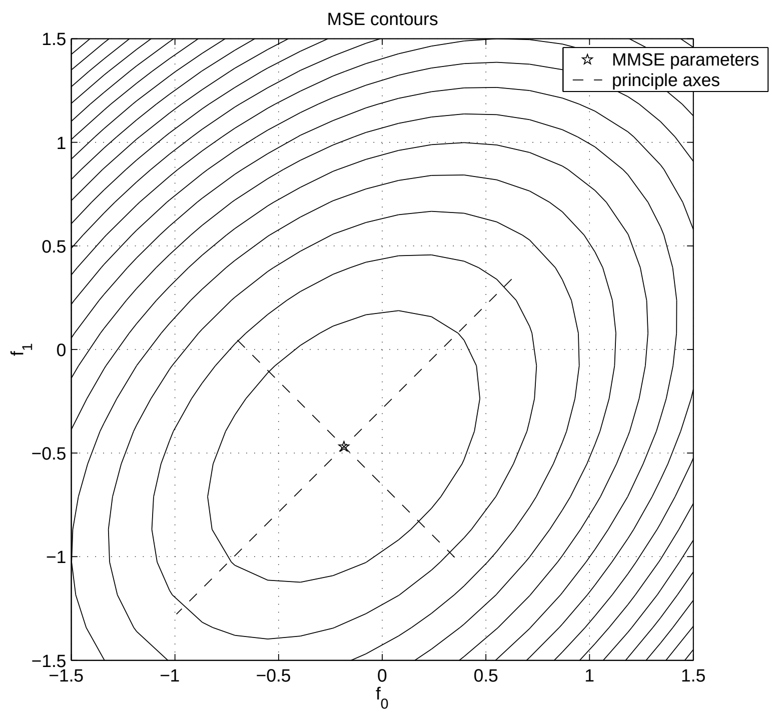

Figure 2: Example MSE cost contours