ECE-894a **Adaptive Filtering** **Autumn 2005**

Homework #2 Sep. 27, 2005

HOMEWORK ASSIGNMENT #2

Due Fri. Oct. 7, 2005 (in class)

1. Consider the block diagram in Fig. 1, where an FIR filter $F^*(z)$ follows an FIR filter $B^*(z)$ driven by white noise $v(n)$ with variance σ_v^2. Assume $B^*(z) = \sum_{k=0}^{K} b_k^* z^{-k}$ and $F^*(z) = \sum_{k=0}^{L} f_k^* z^{-k}$. Is it possible to force $y(n) = v(n-\delta)$ for some delay δ? In other words, is it possible for $F^*(z)$ to invert the effect of $B^*(z)$? This problem is known as "baud spaced equalization of an FIR channel" in the digital communication literature.

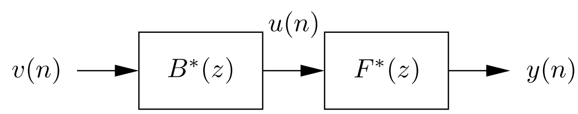

Figure 1: Single channel equalization.

   (a) Using $\mathbf{f} := [f_0, f_1, \ldots, f_L]^t$, the "standard basis vector" $\mathbf{e}_\delta := [0, \ldots, 0, 1, 0, \ldots, 0]^t$ (where the 1 is in the δ^{th} position), $\mathbf{v}(n) := [v(n), v(n-1), \ldots, v(n-K-L)]^t$, and convolution matrix $\mathbf{B}$ (defined as in the lecture), write a vector/matrix equation for the error $e(n) := y(n) - v(n-\delta)$.

   (b) Derive an expression for $\mathrm{E}\{|e(n)|^2\}$ involving $\mathbf{B}, \mathbf{f}, \mathbf{e}_\delta$ and σ_v^2 and the ℓ_2 norm (i.e., $\|\mathbf{x}\|_2^2 := \sum_n |x_n|^2 = \mathbf{x}^H\mathbf{x}$).

   (c) Assuming $b_0 \neq 0 \neq b_K$ and $K > 0$, can we achieve $\mathrm{E}\{|e(n)|^2\} = 0$ for some choice of $\mathbf{f}$ and δ? Prove your claim.

   (d) Repeat (c) for the trivial channel $B^*(z) = z^{-\delta}$.

2. Now consider the two-channel model in Fig. 2. As in the previous problem, $B^*(z)$ and $C^*(z)$ are length-$(K+1)$ FIR filters, $F^*(z)$ and $G^*(z)$ are length-$(L+1)$ FIR filters, and $v(n)$ is variance-σ_w^2 white noise. Such multi-channel models arise in applications employing multiple sensors/antennas or in so-called "fractionally-spaced" equalizers that are used in modern digital communications. We are interested in determining whether it is possible to force $y(n) = v(n-\delta)$.

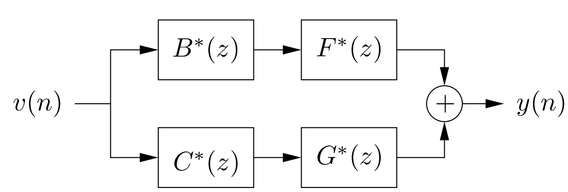

Figure 2: Multiple channel equalization.

The main questions are: Given FIR $B^*(z)$ and $C^*(z)$, can we perfectly recover the input signal (with delay δ) using length-K FIR filters $F^*(z)$ and $G^*(z)$? If so, what conditions must the channels $B^*(z)$ and $C^*(z)$ satisfy?

(a) Using the quantities $\mathbf{f}, \mathbf{g}, \mathbf{B}, \mathbf{C}, \mathbf{e}_\delta$, and $\mathbf{v}(n)$, defined as in the previous problem, write a vector/matrix equation for the error $e(n) := y(n) - v(n-\delta)$. (*Hint:* It may help to stack $\mathbf{f}$ and $\mathbf{g}$ into a single column vector for later parts of this problem.)

(b) Translate the condition $\mathrm{E}\{|e(n)|^2\} = 0$ into a vector/matrix equation involving $[\mathbf{f}^t, \mathbf{g}^t]^t$, $[\mathbf{B}, \mathbf{C}]$, and $\mathbf{e}_\delta$. What conditions on the rank of $[\mathbf{B}, \mathbf{C}]$ are required if we want to achieve $\mathrm{E}\{|e(n)|^2\} = 0$ for *any* $\delta \in \{0, 1, \ldots, L+K\}$? What does this imply about the minimum length of $\mathbf{f}$ & $\mathbf{g}$?

(c) Assuming the rank conditions in part (c) are satisfied, provide equations for the equalizing coefficients $[\mathbf{f}^t, \mathbf{g}^t]^t$ for a fixed choice $\delta \in \{0, 1, \ldots, L+K-1\}$. (*Hint:* use the psuedo-inverse.)

3. The gain of a narrowband multipath communications channel can be modelled by a WSS complex random process $h(n) = h_I(n) + jh_Q(n)$, where $h_I(n)$ and $h_Q(n)$ are the real-valued "in-phase" and "quadrature" components, respectively, and n is a discrete time index. Under "isotropic scattering," the in-phase and quadrature components have temporal correlation specified by

$$\begin{aligned} E[h_I(n)h_I(n-k)] \;=\; E[h_Q(n)h_Q(n-k)] &= \frac{P}{2} J_0\big(2\pi k f_m/F_s\big) \\ E[h_I(n)h_Q(n-k)] &= 0 \;\forall k \end{aligned}$$

where $J_0(\cdot)$ is a zero-order Bessel function of the first kind, P is the total received power, f_m is the ratio of mobile velocity to carrier wavelength, and F_s is the symbol rate.

Our goal is to generate a model of the in-phase component in MATLAB using AR models of order $M = 2, 4, 5$. Assume that $f_m/F_s = 0.1$ and $P = 1$.

(a) Using the Yule-Walker equations and the correlations $r(k) = E[h_I(n)h_I(n-k)]$ specified above, solve for the AR filter coefficients $\{a_1, a_2, \ldots, a_M\}$ and the input variance σ_v^2 for each of the M above. (Hint: use `besselj, toeplitz`.)

(b) Verify each of your designs by generating a random sequence of length $N = 10,000$ satisfying the conditions for $v(n)$ and then filtering it to produce the sequence $u(n)$. (Hint: use `randn, filter`.)

(c) Estimate the autocorrelation of $u(n)$ via

$$\hat{r}_u(k) = \frac{1}{N-k} \sum_{n=0}^{N-1-k} u(n+k)u(n)$$

and plot it along with the desired autocorrelation for $0 \le k \le 200$. Also show a close-up plot of $0 \le k \le 20$. Fig. 3 shows an example for one particular choice of M. (Hint: use `xcorr, subplot, plot, legend, title`.)

Do *not* use any of the built-in Yule-Walker commands. Please turn in your MATLAB code with your homework.

4. You may have noticed that the ability to match the desired autocorrelation sequence in the previous problem was somewhat dissappointing, even for the higher-order AR models. We are now going to repeat the previous design task using the "extended Yule-Walker equations" to see if we can improve the results. The idea is that the standard Yule-Walker equations *exactly* fit the first $M+1$ autocorrelation values but ignore the rest; instead we will attempt an *approximate* least-squares (LS) fit to the first $L+1 \gg M+1$ autocorrelation values.

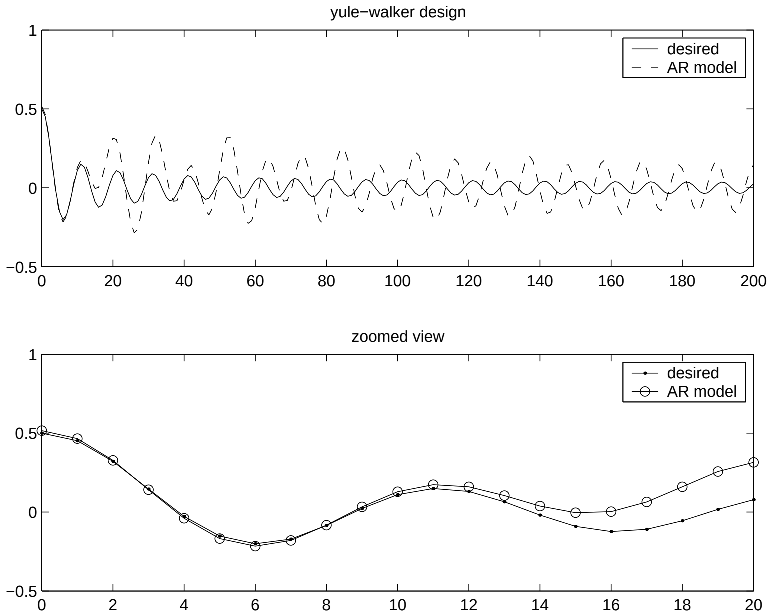

Figure 3: Desired and measured autocorrelations from Yule-Walker AR model.

<u>*Extended Yule-Walker Procedure:*</u>

i. Choose $\mathbf{a} = [a_1, a_2, \ldots, a_M]^t$ as the LS solution to

$$\begin{pmatrix} r(0) & r(1) & \cdots & r(M-1) \\ r^*(1) & r(0) & \cdots & r(M-2) \\ \vdots & & \ddots & \vdots \\ r^*(M-1) & & & r(0) \\ \hline \vdots & & & \vdots \\ r^*(L-1) & & & r^*(L-M) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_M \end{pmatrix} = - \begin{pmatrix} r^*(1) \\ r^*(2) \\ \vdots \\ r^*(M) \\ \hline \vdots \\ r^*(L) \end{pmatrix}$$

which in can be accomplished using the psuedoinverse, i.e., $\mathbf{a} = -\mathbf{R}_L^+ \mathbf{r}_L^*$, where $\mathbf{R}_L$ and $\mathbf{r}_L^*$ are created as above from the *desired* autocorrelation sequence. (Hint: use `toeplitz, pinv`.)

ii. Next we find the *actual* autocorrelation sequence $\{r_u(k)\}$ generated by the AR model. Using the properties

$$\sum_{k=0}^{M} a_k r_u(k-\ell) \;=\; \sum_{k=0}^{M} a_k^* r_u(\ell-k) \;=\; \begin{cases} \sigma_v^2 & \ell = 0 \\ 0 & \ell > 0 \end{cases}$$

we write the system of equations

$$\begin{pmatrix} 1 & a_1 & \cdots & a_M & & & \\ & 1 & a_1 & \cdots & a_M & & \\ & & \ddots & & & \ddots & \\ & & & 1 & a_1 & \cdots & a_M \\ 0 & a_M^* & \cdots & a_1^* & 1 & & \\ \vdots & & \ddots & & & \ddots & \\ 0 & & & a_M^* & \cdots & a_1^* & 1 \end{pmatrix} \begin{pmatrix} r_u(-M) \\ \vdots \\ r_u(-1) \\ r_u(0) \\ r_u(1) \\ \vdots \\ r_u(M) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sigma_v^2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

which yields a unique solution for $\{r_u(-M),\ldots,r_u(M)\}$ when $\{a_k\}$ and σ_v^2 are known. Given these initial values of $r_u(k)$, others can easily be calculated through the recursive filtering operation

$$r_u(\ell) = -\sum_{k=1}^{M} a_k^* r_u(\ell-k), \quad \ell \geq M+1.$$

For now, assume $\sigma_v^2 = 1$ and calculate $\{r_u(k)\big|_{\sigma_v^2=1}$for $k = 0..L\}$. (Hint: use `convmtx, inv`.)

iii. The final step is to select σ_v^2 that best fits $\{r_u(k)\}$ to the desired $\{r(k)\}$. First, realize that the autocorrelation scales linearly with the the input variance, i.e., $r_u(k) = \sigma_v^2 \cdot r_u(k)\big|_{\sigma_v^2=1}$. It follows that the LS optimization problem and solution are:

$$\min_{\sigma_v^2} \sum_{k=0}^{L} \left|\sigma_v^2 \cdot r_u(k)\big|_{\sigma_v^2=1} - r(k)\right|^2 \qquad \Rightarrow \qquad \sigma_v^2\big|_{\text{LS}} = \frac{\sum_{k=0}^{L} r_u(k)\big|_{\sigma_v^2=1} r(k)}{\sum_{k=0}^{L}\left(r_u(k)\big|_{\sigma_v^2=1}\right)^2}.$$

For this problem, generate the same plots as in the problem 4 but using the Extended Yule-Walker procedure outlined above. Assume a fifth order AR model (i.e., $M = 5$) and fitting lengths of $L = 10, 20, 100$. Explain the advantages and disadvantages of choosing $L \gg M$.