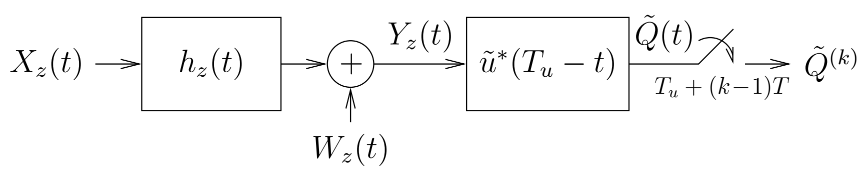
Freq-Selective Stream Demod [Ch. 12]:

Linear binary stream modulation ($i = [m_1, m_2, \ldots, m_{K_b}]$):

$$\begin{aligned}
x_i(t) &= \sum_{k=1}^{K_b} a(m_k) u\big(t-(k-1)T\big) \\
\tilde{x}_i(t) &= x_i(t) * h_z(t), \qquad \tilde{u}(t) \;=\; u(t) * h_z(t) \\
\underline{\hat{I}} &= \arg\max_i \operatorname{Re} \int_0^{T_p+T_h} Y_z(t)\tilde{x}_i^*(t)dt - \frac{\tilde{E}_i}{2} \\
&= \arg\max_i \operatorname{Re} \sum_{k=1}^{K_b} a^*(m_k) \\
&\quad \times \underbrace{\int_0^{T_p+T_h} Y_z(t)\tilde{u}^*\big(t-(k-1)T\big)dt}_{\text{sufficient statistics } \tilde{Q}^{(k)}} - \frac{\tilde{E}_i}{2}
\end{aligned}$$



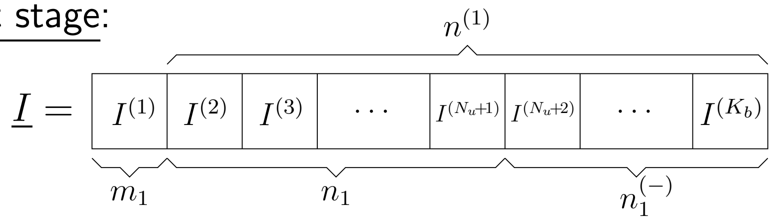
$$\tilde{Q}^{(k)} \;=\; \sum_{l=1}^{K_b} D_z^{(l)} V_{\tilde{u}}\big((k-l)T\big) + \tilde{N}^{(k)}$$

Vector model:

$$\begin{aligned}
[\boldsymbol{G}]_{k,l} &= E_b^{-1} V_{\tilde{u}}\big((k-l)T\big), \quad V_{\tilde{u}}(kT)\Big|_{|k| > \lceil \frac{T_u+T_h}{T} \rceil} = 0 \\
\underline{\tilde{Q}} &= E_b \boldsymbol{G} \underline{D} + \underline{\tilde{N}} \\
\tilde{E}_i &= E_b\, \underline{d}_i^H \boldsymbol{G} \underline{d}_i \\
\underline{\hat{I}} &= \arg\max_i \underbrace{\operatorname{Re} \underline{d}_i^H \underline{\tilde{Q}} - \frac{E_b}{2} \underline{d}_i^H \boldsymbol{G} \underline{d}_i}_{T_i}
\end{aligned}$$

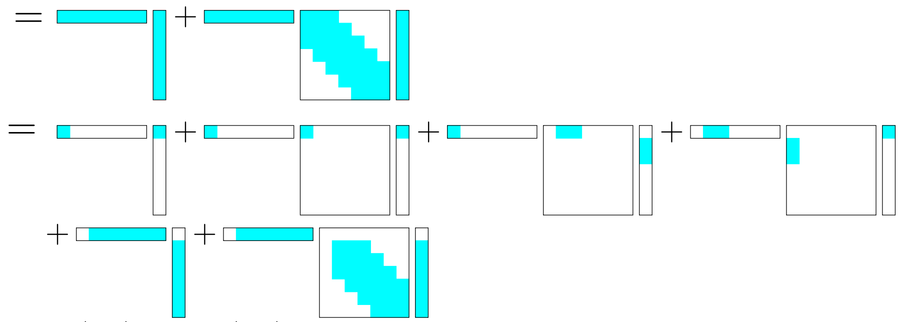
Ungerboeck MLWD:

First stage:



$$T_i = \mathrm{Re}\, \underline{d}_i^H \underline{\tilde{Q}} - \tfrac{E_b}{2} \underline{d}_i^H \boldsymbol{G} \underline{d}_i$$

$T_i =$



$$= T^{(1+)}_{m_1,n_1} + T^{(1-)}_{n^{(1)}}$$

$T^{(1+)}_{m_1,n_1}$ are called “forward metrics.”

Main Idea:

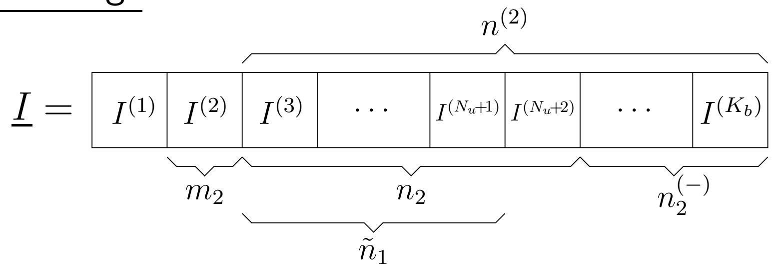
For each $n_1 \in \{0, \ldots, 2^{N_u} - 1\}$, choose the best m_1.

$$\hat{I}^{(1)}_{n_1} = \arg\max_{m_1} T^{(1+)}_{m_1,n_1} \quad \text{“conditional decision”}$$

$$\tilde{T}^{(1)}_{n_1} = \max_{m_1} T^{(1+)}_{m_1,n_1} \quad \text{“conditional metric”}$$

Decide among the n_1 later. . .

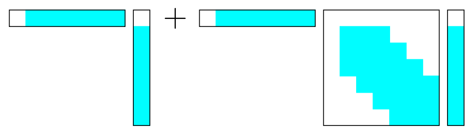
Second stage:



Note that

$$T_i = T^{(1+)}_{m_1,n_1} + T^{(1-)}_{n^{(1)}}$$

$$= T^{(1+)}_{m_1,n_1} +$$



$$= T^{(1+)}_{m_1,n_1} + T^{(2)}_{m_2,n_2} + T^{(2-)}_{n^{(2)}}$$

using the same decoupling idea as before.

Rewrite the metrics:

$$\begin{aligned}\arg\max_i T_i &= \arg \max_{m_1,m_2,n^{(2)}} T^{(1+)}_{m_1,n_1} + T^{(2)}_{m_2,n_2} + T^{(2-)}_{n^{(2)}} \\ &= \arg \max_{m_2,n^{(2)}} \tilde{T}^{(1)}_{n_1} + T^{(2)}_{m_2,n_2} + T^{(2-)}_{n^{(2)}} \\ &= \arg \max_{m_2,n^{(2)}} \underbrace{\tilde{T}^{(1)}_{m_2,\tilde{n}_1} + T^{(2)}_{m_2,n_2}}_{T^{(2+)}_{m_2,n_2}} + T^{(2-)}_{n^{(2)}}\end{aligned}$$

So, for <u>each</u> $n_2 \in \{0, \ldots, 2^{N_u} - 1\}$, choose the best m_2

$$\hat{I}^{(2)}_{n_2} = \arg\max_{m_2} T^{(2+)}_{m_2,n_2}, \qquad \tilde{T}^{(2)}_{n_2} = \max_{m_2} T^{(2+)}_{m_2,n_2}$$

and discard non-surviving paths: $\hat{\underline{I}}^{(2)}_{n_2} = \big[\hat{I}^{(1)}_{\hat{I}^{(2)}_{n_2},\tilde{n}_1} \quad \hat{I}^{(2)}_{n_2}\big]$.

k^{th} stage, (for $2 \le k \le K_b - N_u$):

Can rewrite the metrics as

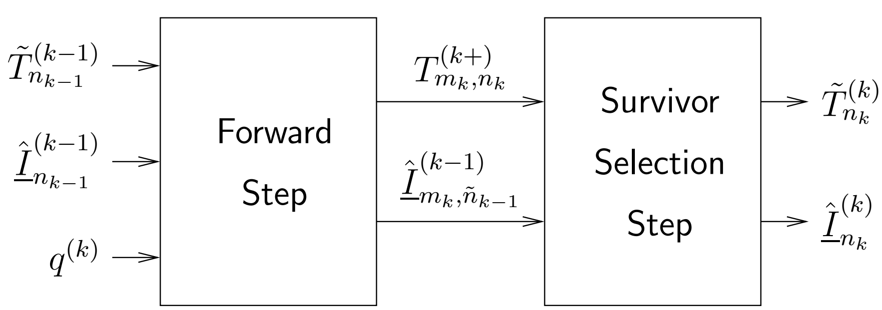
$$\arg\max_i T_i = \arg\max_{m_k, n^{(k)}} \underbrace{\tilde{T}^{(k-1)}_{m_k,\tilde{n}_{k-1}} + T^{(k)}_{m_k,n_k}}_{T^{(k+)}_{m_k,n_k}} + T^{(k-)}_{n^{(k)}}$$

Two essential steps:

1. For each of the 2^{N_u+1} possible values of $\{m_k, n_k\}$, compute the branch metric $T^{(k)}_{m_k,n_k}$ and add it to the corresponding partial survivor metric $\tilde{T}^{(k-1)}_{m_k,\tilde{n}_{k-1}}$.
2. Keep the 2^{N_u} surviving paths:

$$\hat{I}^{(k)}_{n_k} = \arg\max_{m_k} T^{(k+)}_{m_k,n_k}, \quad \underline{\hat{I}}^{(k)}_{n_k} = \big[\underline{\hat{I}}^{(k-1)}_{\hat{I}^{(k)}_{n_k},\tilde{n}_{k-1}} \quad \hat{I}^{(k)}_{n_k}\big]$$

In summary, the k^{th} Ungerboeck recursion looks like



Termination:

- After the $k = (K_b - N_u)^{th}$ stage, grow $\{\tilde{T}^{(k)}_{n_k}\}$ into the surviving cumulative metrics using $[q^{(k+1)}, \ldots, q^{(K_b)}]$.
- Choose the largest of these 2^{N_u} cumulative metrics.

Overall complexity of MLWD: $\mathcal{O}(K_b N_u 2^{N_u+1})$.

Example of Ungerboeck MLWD:

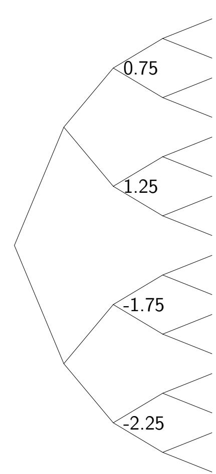
- BPSK, $K_b = 4$, $u(t) = \sqrt{\frac{1}{T}}\, 1_{[0,T]}(t)$
- $h_z(t) = \sqrt{2}\delta(t) + \sqrt{2}e^{j\frac{\pi}{3}}\delta(t-T),\ E_{\tilde{u}} = 1$
- $\boldsymbol{G} = \begin{pmatrix} 1 & 0.5e^{j\frac{\pi}{3}} & 0 & 0 \\ 0.5e^{-j\frac{\pi}{3}} & 1 & 0.5e^{j\frac{\pi}{3}} & 0 \\ 0 & 0.5e^{-j\frac{\pi}{3}} & 1 & 0.5e^{j\frac{\pi}{3}} \\ 0 & 0 & 0.5e^{-j\frac{\pi}{3}} & 1 \end{pmatrix}$
- $\underline{\tilde{q}} = [1.5 + j0.5,\ 0.3 + j0.7,\ 1.2 + j0.1,\ -0.9 + j0.1]$
- $N_u = 1 \quad \Rightarrow \quad 2^{N_u} = 2 \quad \Rightarrow \quad n_k \in \{0, 1\}$

<u>Time 1:</u>

$$T^{(1+)}_{m_1,n_1} = \mathrm{Re}\big[d^{(1)*}_{m_1}\tilde{q}^{(1)}\big] - \frac{E_{\tilde{u}}}{2}\big[|d^{(1)}_{m_1}|^2 + d^{(1)*}_{m_1}q_{1,2}d^{(2)}_{n_1} + d^{(2)*}_{n_1}q_{2,1}d^{(1)}_{m_1}\big]$$

$$\begin{aligned} T^{(1+)}_{0,0} &= 0.75, \\ T^{(1+)}_{1,0} &= -1.75, \\ T^{(1+)}_{0,1} &= 1.25, \\ T^{(1+)}_{1,1} &= -2.25, \end{aligned}$$

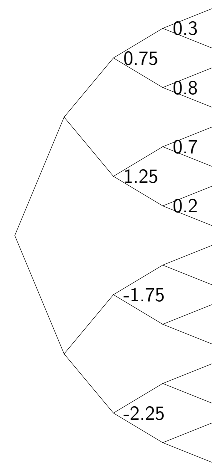
$$\begin{aligned} \tilde{T}^{(1)}_0 &= 0.75, \qquad \hat{I}^{(1)}_0 = 0 \\ \tilde{T}^{(1)}_1 &= 1.25, \qquad \hat{I}^{(1)}_1 = 0 \end{aligned}$$



<u>Time 2:</u> (note: $n_1 = m_2$)

$$T^{(2)}_{m_2,n_2} = \mathrm{Re}\big[d^{(2)*}_{m_2}\tilde{q}^{(2)}\big] - \frac{E_{\tilde{u}}}{2}\big[|d^{(2)}_{m_2}|^2 + d^{(3)}_{n_2}d^{(2)*}_{m_2}q_{2,3} + d^{(2)}_{m_2}d^{(3)*}_{n_2}q_{3,2}\big]$$

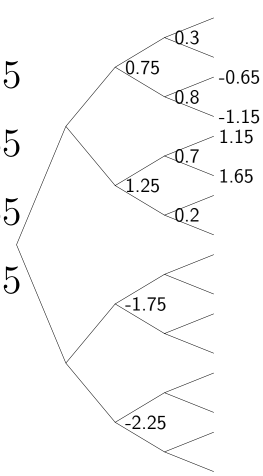
$$\begin{aligned}
T^{(2)}_{0,0} &= -0.45, & T^{(2+)}_{0,0} &= T^{(2)}_{0,0} + \tilde{T}^{(1)}_0 = 0.3 \\
T^{(2)}_{1,0} &= -0.55, & T^{(2+)}_{1,0} &= T^{(2)}_{1,0} + \tilde{T}^{(1)}_1 = 0.7 \\
T^{(2)}_{0,1} &= 0.05, & T^{(2+)}_{0,1} &= T^{(2)}_{0,1} + \tilde{T}^{(1)}_0 = 0.8 \\
T^{(2)}_{1,1} &= -1.05, & T^{(2+)}_{1,1} &= T^{(2)}_{1,1} + \tilde{T}^{(1)}_1 = 0.2 \\
\tilde{T}^{(2)}_0 &= 0.7, & \underline{\hat{I}}^{(2)}_0 &= [0,1] \\
\tilde{T}^{(2)}_1 &= 0.8, & \underline{\hat{I}}^{(2)}_1 &= [0,0]
\end{aligned}$$



<u>Time 3:</u> (note: $n_2 = m_3$)

$$T^{(3)}_{m_3,n_3} = \mathrm{Re}\big[d^{(3)*}_{m_3}\tilde{q}^{(3)}\big] - \frac{E_{\tilde{u}}}{2}\big[|d^{(3)}_{m_3}|^2 + d^{(4)}_{n_3}d^{(3)*}_{m_3}q_{3,4} + d^{(3)}_{m_3}d^{(4)*}_{n_3}q_{4,3}\big]$$

$$\begin{aligned}
T^{(3)}_{0,0} &= 0.45, & T^{(3+)}_{0,0} &= T^{(3)}_{0,0} + \tilde{T}^{(2)}_0 = 1.15 \\
T^{(3)}_{1,0} &= -1.45, & T^{(3+)}_{1,0} &= T^{(3)}_{1,0} + \tilde{T}^{(2)}_1 = -0.65 \\
T^{(3)}_{0,1} &= 0.95, & T^{(3+)}_{0,1} &= T^{(3)}_{0,1} + \tilde{T}^{(2)}_0 = 1.65 \\
T^{(3)}_{1,1} &= -1.95, & T^{(3+)}_{1,1} &= T^{(3)}_{1,1} + \tilde{T}^{(2)}_1 = -1.15 \\
\tilde{T}^{(3)}_0 &= 1.15, & \underline{\hat{I}}^{(3)}_0 &= [0,1,0] \\
\tilde{T}^{(3)}_1 &= 1.65, & \underline{\hat{I}}^{(3)}_1 &= [0,1,0]
\end{aligned}$$



Time 4: (termination)

$$T_{m_4}^{(4)} = \mathrm{Re}\big[d_{m_4}^{(4)*}\tilde{q}^{(4)}\big] - \frac{E_{\tilde{u}}}{2}\big[|d_{m_4}^{(4)}|^2\big]$$

$$T_0^{(4)} = -1.4, \quad T_{[0100]} = T_0^{(4)} + \tilde{T}_0^{(3)} = -0.25$$
$$T_1^{(4)} = 0.4, \quad T_{[0101]} = T_1^{(4)} + \tilde{T}_1^{(3)} = 2.05$$

$$\Rightarrow \underline{\hat{I}} = [0, 1, 0, 1]$$