Error Analysis using the Trellis [Ch. 14]:

Main points:

- We've seen how the union bound approximates the word error probability of MLWD.
- Now we will see how the trellis structure can be used to tighten the union bound via "simple error events."

Recall MLWD:

$$\begin{aligned}\hat{\underline{I}} &= \arg\max_i T_i \quad \text{for} \quad T_i = \mathrm{Re}\,\sqrt{E_b}\,\tilde{\underline{d}}_i^H \underline{Q} - \tfrac{E_b}{2}\|\tilde{\underline{d}}_i\|^2 \\ &= \arg\max_i \tilde{T}_i \quad \text{for} \quad \tilde{T}_i \stackrel{\Delta}{=} -\|\underline{Q} - \sqrt{E_b}\tilde{\underline{d}}_i\|^2\end{aligned}$$

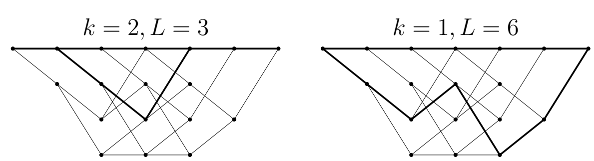
The error event $\{\hat{\underline{I}} = i | \underline{I} = j\}$ implies that $\{\tilde{T}_{i|j} > \tilde{T}_{j|j}\}$.

A tightened union bound:

- We say $\{\hat{\underline{I}} = i | \underline{I} = j\}$ is a *simple error event of length* L *at time* k when the edges obey

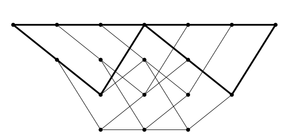
$$\begin{cases} s_i^{(l)} \neq s_j^{(l)} & l \in \{k, \ldots, k+L-1\} \stackrel{\Delta}{=} \mathcal{M} \\ s_i^{(l)} = s_j^{(l)} & l \notin \mathcal{M} \end{cases}$$

Examples:



- When $\{\hat{\underline{I}} = i | \underline{I} = j\}$ consists of several simple error events, we say that it is a *compound error event*.

Example:



Idea: Remove compound error events from the union bound!

- Consider the compound error event $\{\hat{\underline{I}} = i | \underline{I} = j\}$ defined by the edge-error index sets $\mathcal{M}_1$ and $\mathcal{M}_2$, each of which defines a simple error event. ($\mathcal{M}_1 \cap \mathcal{M}_2 = \emptyset$.)
- This event yields decoupled conditional-ML metrics:

$$\begin{aligned}
\tilde{T}_{i|j} &= \sum_{l\in\mathcal{M}_1} \left|Q^{(l)} - \sqrt{E_b} d_i^{(l)}\right|^2 + \sum_{l\in\mathcal{M}_2} \left|Q^{(l)} - \sqrt{E_b} d_i^{(l)}\right|^2 \\
&\quad + \sum_{l\notin\mathcal{M}_1\cup\mathcal{M}_2} \left|Q^{(l)} - \sqrt{E_b} d_i^{(l)}\right|^2 \quad \text{given } \underline{I} = j. \\
&= \tilde{T}_{i|j}^{\mathcal{M}_1} + \tilde{T}_{i|j}^{\mathcal{M}_2} + \tilde{T}_{i|j}^{\overline{\mathcal{M}_2\cup\mathcal{M}_1}} \\
\tilde{T}_{j|j} &= \tilde{T}_{j|j}^{\mathcal{M}_1} + \tilde{T}_{j|j}^{\mathcal{M}_2} + \tilde{T}_{j|j}^{\overline{\mathcal{M}_2\cup\mathcal{M}_1}}
\end{aligned}$$

- $\tilde{T}_{i|j}^{\overline{\mathcal{M}_2\cup\mathcal{M}_1}} = \tilde{T}_{j|j}^{\overline{\mathcal{M}_2\cup\mathcal{M}_1}}$ due to common symbols, so that

$$\begin{aligned}
& \{\tilde{T}_{i|j} > \tilde{T}_{j|j}\} \\
\Leftrightarrow\ & \{\tilde{T}_{i|j}^{\mathcal{M}_1} + \tilde{T}_{i|j}^{\mathcal{M}_2} > \tilde{T}_{j|j}^{\mathcal{M}_1} + \tilde{T}_{j|j}^{\mathcal{M}_2}\} \\
\Leftrightarrow\ & \{\tilde{T}_{i|j}^{\mathcal{M}_1} > \tilde{T}_{j|j}^{\mathcal{M}_1}\} \text{ and/or } \{\tilde{T}_{i|j}^{\mathcal{M}_2} > \tilde{T}_{j|j}^{\mathcal{M}_2}\} \\
\Leftrightarrow\ & \{\tilde{T}_{i|j}^{\mathcal{M}_1} > \tilde{T}_{j|j}^{\mathcal{M}_1}\} \cup \{\tilde{T}_{i|j}^{\mathcal{M}_2} > \tilde{T}_{j|j}^{\mathcal{M}_2}\}
\end{aligned}$$

- Note that, in the PWE expression, the compound event is already represented by these two simple events:

$$\Pr(\hat{\underline{I}} \neq j | \underline{I} = j) = \Pr\left(\bigcup_{\substack{i\neq j\\ i=0}}^{2^{K_b}-1} \{\tilde{T}_{i|j} > \tilde{T}_{j|j}\}\right) \leq \underbrace{\sum_{\substack{i\neq j\\ i=0}}^{2^{K_b}-1} \Pr(\tilde{T}_{i|j} > \tilde{T}_{j|j})}_{\text{old union bound}}$$

So remove the compound event from the union bound!

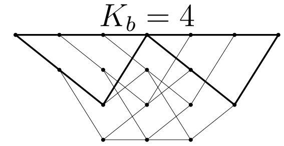
In summary, a tighter union bound can be obtained by removing all compound error events.

$$\Pr(\hat{\underline{I}} \neq \underline{I}) = \sum_{j=0}^{2^{K_b}-1} \Pr(\hat{\underline{I}} \neq j | \underline{I} = j) \Pr(\underline{I} = j)$$

$$\Pr(\hat{\underline{I}} \neq j | \underline{I} = j) \leq \sum_{k=1}^{K_b} \sum_{i \in \tilde{\Omega}_j^{(k)}} \Pr(\tilde{T}_{i|j} > \tilde{T}_{j|j})$$

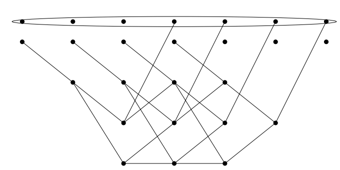
$$\tilde{\Omega}_j^{(k)} \triangleq \text{indices of words forming simple errors with } \underline{I} = j \text{ starting at time } k.$$

In example on right, there is only one compound error event for $j = 0$. Bound tightening will be more significant for larger K_b.



To help enumerate the simple error events, a *modified trellis* can be used. There, an *absorbing state* is added to each trellis stage to facilitate completion of simple error events.

The modified trellis for $\underline{I} = j = 0$ is: (Need a different trellis for each j.)



Recalling that $\Pr\big(\tilde{T}_{i|j} > \tilde{T}_{j|j}\big) = \frac{1}{2}\operatorname{erfc}\Big(\sqrt{\frac{\Delta_E(i,j)}{4N_o}}\Big)$, we see that the modified trellis also helps in computing $\Delta_E(i,j)$ for each simple error event $\{\hat{\underline{I}} = i | \underline{I} = j\}$. Specifically, we have

$$\Delta_E(i,j) = E_b \sum_{l \in \mathcal{M}(i,j)} \big|\tilde{d}_i^{(l)} - \tilde{d}_j^{(l)}\big|^2$$

where $\mathcal{M}(i,j)$ contains the time indices of the error path.

Large Frame Error Analysis:

- Error analysis that requires enumeration of all paths through a modified trellis is feasible only for small K_b. So how is error analysis accomplished for large K_b?
- Ignore start/finish of trellis (i.e., $1 \ll k \ll N_f$). Then, $\Delta_E(i,j)$ for $i \in \tilde{\Omega}_j^{(k)}$ (and any fixed j) is insensitive to k, so consider an arbitrary k. This gives the bound:

$$\Pr\big(\hat{\underline{I}} \neq \underline{I} | \underline{I} = j\big) \quad \leq \quad K_b \sum_{i \in \tilde{\Omega}_j^{(k)}} \tfrac{1}{2} \operatorname{erfc}\Big(\sqrt{\tfrac{\Delta_E(i,j)}{4N_o}}\Big)$$

  These simple errors $\tilde{\Omega}_j^{(k)}$ starting at a single k are called *first error events*.
- We will develop techniques to evaluate this bound.

Two approaches to quantify $\frac{1}{2} \operatorname{erfc}\Big(\sqrt{\frac{\Delta_E(i,j)}{4N_o}}\Big)$ for $i \in \tilde{\Omega}_j^{(k)}$ are given below. Both use $\Delta_E^{(l)}(i,j) \triangleq E_b \big|\tilde{d}_i^{(l)} - \tilde{d}_j^{(l)}\big|^2$:

1. Chernoff bound:

$$\begin{aligned} \tfrac{1}{2} \operatorname{erfc}\Big(\sqrt{\tfrac{\Delta_E(i,j)}{4N_o}}\Big) \quad &\leq \quad \tfrac{1}{2} \exp\Big(\tfrac{-\Delta_E(i,j)}{4N_o}\Big) \\ &= \quad \tfrac{1}{2} \prod_{l=1}^{L(i,j)} \exp\Big(\tfrac{-\Delta_E^{(k+l-1)}(i,j)}{4N_o}\Big) \end{aligned}$$

2. Craig's form (exact):

$$\begin{aligned} \tfrac{1}{2} \operatorname{erfc}\Big(\sqrt{\tfrac{\Delta_E(i,j)}{4N_o}}\Big) \quad &= \quad \tfrac{1}{\pi} \int_0^{\frac{\pi}{2}} \exp\Big(\tfrac{-\Delta_E(i,j)}{4N_o \cos^2(\tau)}\Big) d\tau \\ &= \quad \tfrac{1}{\pi} \int_0^{\frac{\pi}{2}} \prod_{l=1}^{L(i,j)} \exp\Big(\tfrac{-\Delta_E^{(k+l-1)}(i,j)}{4N_o \cos^2(\tau)}\Big) d\tau \end{aligned}$$

We choose Craig's form and thus obtain the union bound

$$\Pr\big(\underline{\hat{I}} \neq \underline{I} | \underline{I} = j\big) \;\leq\; K_b \sum_{i\in\tilde{\Omega}_j^{(k)}} \tfrac{1}{2}\,\mathrm{erfc}\Big(\sqrt{\tfrac{\Delta_E(i,j)}{4N_o}}\Big)$$

$$= \frac{K_b}{\pi}\int_0^{\frac{\pi}{2}} \sum_{i\in\tilde{\Omega}_j^{(k)}} \prod_{l=1}^{L(i,j)} \exp\Big(\tfrac{-\Delta_E^{(k+l-1)}(i,j)}{4N_o\cos^2(\tau)}\Big)d\tau$$

$$= \frac{K_b}{\pi}\int_0^{\frac{\pi}{2}} \sum_{L=2}^{\infty}\Big[\sum_{i\in\tilde{\Omega}_j^{(k)}(L)} \prod_{l=1}^{L} \exp\Big(\tfrac{-\Delta_E^{(k+l-1)}(i,j)}{4N_o\cos^2(\tau)}\Big)\Big]d\tau$$

where $\tilde{\Omega}_j^{(k)}(L)$ denotes the set of simple error events (relative to $\underline{I} = j$) <u>of length L</u> starting at time k.

Next we evaluate the term in brackets for fixed values of L.

First consider simple errors of length $L = 2$. There,

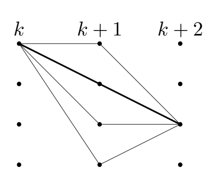
$$\sum_{i\in\tilde{\Omega}_j^{(k)}(2)} \prod_{l=1}^{2} \exp\Big(\tfrac{-\Delta_E^{(k+l-1)}(i,j)}{4N_o\cos^2(\tau)}\Big)$$

$$= \sum_{i\in\tilde{\Omega}_j^{(k)}(L)} \exp\Big(\tfrac{-\Delta_E^{(k)}(i,j)}{4N_o\cos^2(\tau)}\Big)\exp\Big(\tfrac{-\Delta_E^{(k+1)}(i,j)}{4N_o\cos^2(\tau)}\Big)$$

$$= \sum_{\substack{i_s\neq\sigma_j^{(k+1)}\\ i_s=1}}^{N_s} \underbrace{\exp\Big(\tfrac{-\Delta_\sigma\big(\sigma_j^{(k)},i_s,\sigma_j^{(k)},\sigma_j^{(k+1)}\big)}{4N_o\cos^2(\tau)}\Big)}_{\big[\underline{S}_{\mathsf{gb},j}^{(k)}(\tau)\big]_{i_s}} \underbrace{\exp\Big(\tfrac{-\Delta_\sigma\big(i_s,\sigma_j^{(k+2)},\sigma_j^{(k+1)},\sigma_j^{(k+2)}\big)}{4N_o\cos^2(\tau)}\Big)}_{\big[\underline{S}_{\mathsf{bg},j}^{(k+1)}(\tau)\big]_{i_s}}$$

$= \underline{S}_{\mathsf{gb},j}^{(k)}(\tau)\underline{S}_{\mathsf{bg},j}^{(k+1)}(\tau)^T$ using length-(N_s-1) row vectors.

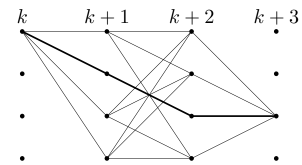
Example of length-2 simple errors for a 4-state trellis: (good → bad → good)



$\underline{S}_{\mathsf{gb}}$: "good→bad",
$\underline{S}_{\mathsf{bg}}$: "bad→good".

Next consider simple errors of length $L = 3$.

Example of length-3 simple errors for a 4-state trellis: (good→bad→bad→good)



We use transition matrix $\boldsymbol{S}^{(k+1)}_{\mathsf{b},j}(\tau)$ to describe transitions among the "bad" states. For the example above,

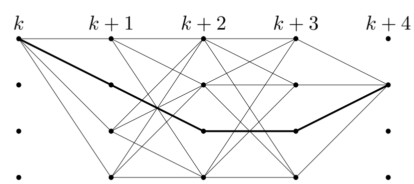
$$\boldsymbol{S}^{(k+1)}_{\mathsf{b},j}(\tau) = \begin{pmatrix} \exp\left(\frac{-\Delta_\sigma(1,1,2,3)}{4N_o\cos^2(\tau)}\right) & \exp\left(\frac{-\Delta_\sigma(1,2,2,3)}{4N_o\cos^2(\tau)}\right) & \exp\left(\frac{-\Delta_\sigma(1,4,2,3)}{4N_o\cos^2(\tau)}\right) \\ \exp\left(\frac{-\Delta_\sigma(3,1,2,3)}{4N_o\cos^2(\tau)}\right) & \exp\left(\frac{-\Delta_\sigma(3,2,2,3)}{4N_o\cos^2(\tau)}\right) & \exp\left(\frac{-\Delta_\sigma(3,4,2,3)}{4N_o\cos^2(\tau)}\right) \\ \exp\left(\frac{-\Delta_\sigma(4,1,2,3)}{4N_o\cos^2(\tau)}\right) & \exp\left(\frac{-\Delta_\sigma(4,2,2,3)}{4N_o\cos^2(\tau)}\right) & \exp\left(\frac{-\Delta_\sigma(4,4,2,3)}{4N_o\cos^2(\tau)}\right) \end{pmatrix}$$

Then

$$\sum_{i\in\tilde{\Omega}^{(k)}_j(3)} \prod_{l=1}^{3} \exp\left(\frac{-\Delta_E^{(k+l-1)}(i,j)}{4N_o\cos^2(\tau)}\right) = \underline{S}^{(k)}_{\mathsf{gb},j}(\tau)\,\boldsymbol{S}^{(k+1)}_{\mathsf{b},j}(\tau)\,\underline{S}^{(k+2)}_{\mathsf{bg},j}(\tau)^T$$

Next consider simle errors of length $L = 4$.

Example of length-4 simple errors for a 4-state trellis:



In this case

$$\sum_{i\in\tilde{\Omega}^{(k)}_j(4)} \prod_{l=1}^{4} \exp\left(\frac{-\Delta_E^{(k+l-1)}(i,j)}{4N_o\cos^2(\tau)}\right)$$
$$= \underline{S}^{(k)}_{\mathsf{gb},j}(\tau)\,\boldsymbol{S}^{(k+1)}_{\mathsf{b},j}(\tau)\,\boldsymbol{S}^{(k+2)}_{\mathsf{b},j}(\tau)\,\underline{S}^{(k+3)}_{\mathsf{bg},j}(\tau)^T$$

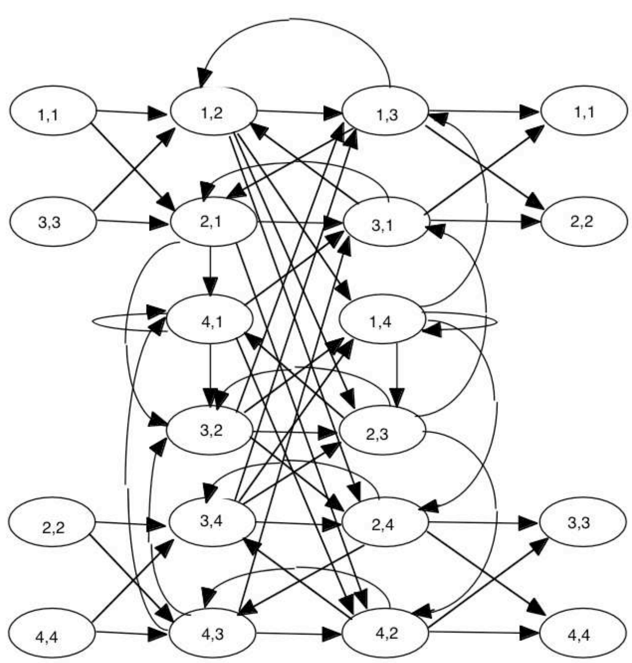
It should now be easy to see what happens for larger L.

This is inconvenient because the expression depends on the true path index, j, and the time of the first error event, k! We can get around this problem using "product states"...

Consider the *product state* $\underline{\sigma}_{i,j}^{(k)} \triangleq \left(\sigma_i^{(k)}, \sigma_j^{(k)}\right)$.

- A sequence of product states measures the deviation between paths i and j.
- There are N_s "good" product states, $\{(i_s, i_s)\}_{i_s=1}^{N_s}$, and $N_s^2 - N_s$ "bad" ones, $\{(i_s, j_s) : j_s \neq i_s\}_{i_s=1}^{N_s}$.
- A length-L first error event will
  1. diverge from a good product state at time k,
  2. transition among the bad product states,
  3. get absorbed by a good product state at time $k + L$.

Product-state diagram for the HCV code:



To describe product-state transitions, we construct matrices

$$\boldsymbol{S}_{\mathsf{gb}}(\tau) \in \mathbb{R}^{N_s \times (N_s^2 - N_s)} \qquad \text{good}\rightarrow\text{bad}$$
$$\boldsymbol{S}_{\mathsf{b}}(\tau) \in \mathbb{R}^{(N_s^2 - N_s) \times (N_s^2 - N_s)} \qquad \text{bad}\rightarrow\text{bad}$$
$$\boldsymbol{S}_{\mathsf{bg}}(\tau) \in \mathbb{R}^{(N_s^2 - N_s) \times N_s} \qquad \text{bad}\rightarrow\text{good}$$

with elements of the form: $\exp\Big(\frac{-\Delta_\sigma(i_s,j_s,k_s,l_s)}{4N_o\cos^2(\tau)}\Big)$. Note the lack of dependance on time k! These lead to

$$\big[\boldsymbol{S}_{\mathsf{gb}}(\tau)\boldsymbol{S}_{\mathsf{b}}^{L-2}(\tau)\boldsymbol{S}_{\mathsf{bg}}(\tau)\big]_{k_s,i_s} = \sum_{j\in\Omega_{k_s,i_s}^{(k,k+L)}} \sum_{i\in\tilde{\Omega}_j^{(k)}(L)} \prod_{l=1}^{L} \exp\Big(\frac{-\Delta_E^{(k+l-1)}(i,j)}{4N_o\cos^2(\tau)}\Big)$$

Note the contribution from several true paths $j \in \Omega_{k_s,i_s}^{(k,k+L)}$.

We will now average the conditional union bound over j:

$$\Pr\big(\underline{\hat{I}} \neq \underline{I} | \underline{I} = j\big) \leq \frac{K_b}{\pi}\int_0^{\frac{\pi}{2}} \sum_{L=2}^{\infty} \sum_{i\in\tilde{\Omega}_j^{(k)}(L)} \prod_{l=1}^{L} \exp\Big(\frac{-\Delta_E^{(k+l-1)}(i,j)}{4N_o\cos^2(\tau)}\Big) d\tau$$

For any time k, can reason that $\sigma_j^{(k)}$ is uniformly distributed with probability $\frac{1}{N_s}$. Furthermore, there are 2^L equally likely length-L true paths emanating from $\sigma_j^{(k)}$. Thus,

$$\Pr\big(\underline{\hat{I}} \neq \underline{I}\big) \leq \frac{K_b}{\pi}\int_0^{\frac{\pi}{2}} \sum_{L=2}^{\infty} \frac{1}{N_s 2^L} \sum_{k_s=1}^{N_s}\sum_{i_s=1}^{N_s} \underbrace{\sum_{j\in\Omega_{k_s,i_s}^{(k,k+L)}} \sum_{i\in\tilde{\Omega}_j^{(k)}(L)} \prod_{l=1}^{L} \exp\Big(\frac{-\Delta_E^{(k+l-1)}(i,j)}{4N_o\cos^2(\tau)}\Big)}_{\left[\boldsymbol{S}_{\mathsf{gb}}(\tau)\boldsymbol{S}_{\mathsf{b}}^{L-2}(\tau)\boldsymbol{S}_{\mathsf{bg}}(\tau)\right]_{k_s,i_s}} d\tau$$

Continuing with the averaged union bound...

$$
\begin{aligned}
&\Pr(\underline{\hat{I}} \neq \underline{I}) \\
&\leq \frac{K_b}{\pi} \int_0^{\frac{\pi}{2}} \sum_{L=2}^{\infty} \frac{1}{N_s 2^L} \sum_{k_s=1}^{N_s} \sum_{i_s=1}^{N_s} \left[\boldsymbol{S}_{\mathsf{gb}}(\tau) \boldsymbol{S}_{\mathsf{b}}^{L-2}(\tau) \boldsymbol{S}_{\mathsf{bg}}(\tau)\right]_{k_s,i_s} d\tau \\
&= \frac{K_b}{4N_s\pi} \int_0^{\frac{\pi}{2}} \sum_{L=2}^{\infty} \frac{1}{2^{L-2}} \underline{1}_{N_s}^T \boldsymbol{S}_{\mathsf{gb}}(\tau) \boldsymbol{S}_{\mathsf{b}}^{L-2}(\tau) \boldsymbol{S}_{\mathsf{bg}}(\tau) \underline{1}_{N_s} \, d\tau \\
&= \frac{K_b}{4N_s\pi} \int_0^{\frac{\pi}{2}} \underline{1}_{N_s}^T \boldsymbol{S}_{\mathsf{gb}}(\tau) \left[\sum_{L=2}^{\infty} \left(\frac{1}{2}\boldsymbol{S}_{\mathsf{b}}(\tau)\right)^{L-2}\right] \boldsymbol{S}_{\mathsf{bg}}(\tau) \underline{1}_{N_s} \, d\tau \\
&= \frac{K_b}{4N_s\pi} \int_0^{\frac{\pi}{2}} \underline{1}_{N_s}^T \boldsymbol{S}_{\mathsf{gb}}(\tau) \left(\boldsymbol{I}_{N_s^2-N_s} - \frac{1}{2}\boldsymbol{S}_{\mathsf{b}}(\tau)\right)^{-1} \boldsymbol{S}_{\mathsf{bg}}(\tau) \underline{1}_{N_s} \, d\tau
\end{aligned}
$$

Can evaluate this integral numerically.