MIDTERM SOLUTIONS

1. For symbol $d_i \in \mathbb{C}$ where $i \in \{0, \ldots, 2^{K_b} - 1\}$, define

$$\begin{aligned} x_i(t) &:= d_i u(t) \\ \tilde{u}(t) &:= \int h_z(\tau) u(t-\tau) d\tau \;=\; \sum_k h_k u(t-\tau_k) \\ \tilde{x}_i(t) &:= d_i \tilde{u}(t) \end{aligned}$$

Note that, since we always use E_b to denote the energy per *received* bit, it makes no sense to put $\sqrt{E_b}$ in the $x_i(t)$ expression. Assuming that j is the transmitted symbol index, the receiver observes

$$Y_z(t) = \tilde{x}_j(t) + W_z(t)$$

with complex white Gaussian process $W_z(t)$ of spectral density N_o. The MLWD is known to take the form

$$\hat{I} = \arg\max_i \mathrm{Re}\left[\int Y_z(t) \tilde{x}_i^*(t) dt\right] - \frac{1}{2}\tilde{E}_i$$

where $\tilde{E}_i := \int |\tilde{x}_i(t)| dt = |d_i|^2 E_{\tilde{u}}$. If we define

$$\begin{aligned} Q &:= \int Y_z(t) \tilde{u}^*(t) dt \\ &= \sum_k h_k^* \int Y_z(t) u^*(t-\tau_k) dt \end{aligned}$$

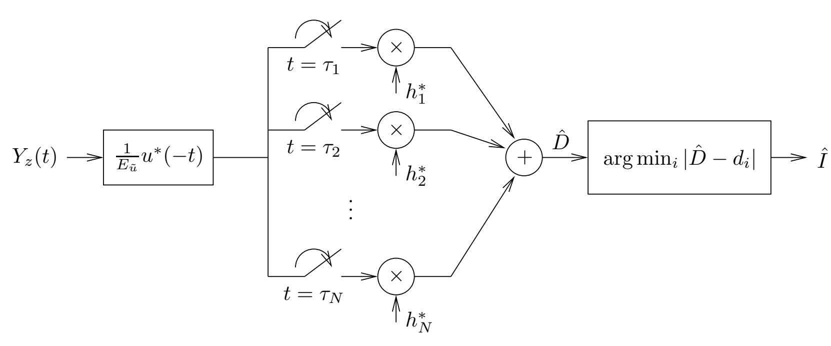
then

$$\begin{aligned} \hat{I} &= \arg\max_i \mathrm{Re}\left[d_i^* Q\right] - \frac{1}{2}|d_i|^2 E_{\tilde{u}} \\ &= \arg\min_i \left| d_i - \frac{Q}{E_{\tilde{u}}} \right|^2 \end{aligned}$$

Note that $\hat{D} := \frac{Q}{E_{\tilde{u}}}$ can be computed by summing the sampled outputs of a single filter:

$$\hat{D} = \sum_k h_k^* \left[Y_z(t) * \frac{u^*(-t)}{E_{\tilde{u}}} \right]_{t=\tau_k}$$

A block diagram summarizing the MLWD appears below.



2. When the true symbol index is j, we have

$$\begin{aligned}
Y_z(t) &= d_j\tilde{u}(t) + W_z(t) \\
\hat{D} &= \frac{1}{E_{\tilde{u}}}\int Y_z(t)\tilde{u}^*(t)dt \\
&= d_j \underbrace{\frac{1}{E_{\tilde{u}}}\int \tilde{u}(t)\tilde{u}^*(t)dt}_{=1} + \underbrace{\frac{1}{E_{\tilde{u}}}\int W_z(t)\tilde{u}^*(t)dt}_{N_z}
\end{aligned}$$

where N_z is zero-mean complex Gaussian with variance

$$\begin{aligned}
\sigma^2_{N_z} &= \mathrm{E}\left\{\frac{1}{E_{\tilde{u}}}\int W_z(t_1)\tilde{u}^*(t_1)dt_1 \times \frac{1}{E_{\tilde{u}}}\int W_z(t_2)\tilde{u}^*(t_2)dt_2\right\} \\
&= \frac{1}{E^2_{\tilde{u}}}\int N_o\delta(\tau)\tilde{u}^*(t)\tilde{u}^*(t-\tau)d\tau \\
&= \frac{N_o}{E_{\tilde{u}}}
\end{aligned}$$

With Gray-mapping, the real component of d_i carries one bit and the imaginary component of d_i carries the other. Since the noise is (circular) complex Gaussian, we can decouple the detection problem into two single-bit detection problems:

$$\begin{aligned}
\mathrm{Re}\,\hat{D} &= \mathrm{Re}\,d_i + \mathrm{Re}\,N_z \\
\mathrm{Im}\,\hat{D} &= \mathrm{Im}\,d_i + \mathrm{Im}\,N_z
\end{aligned}$$

where $\mathrm{Re}\,N_z$ and $\mathrm{Im}\,N_z$ are independent, zero-mean, real Gaussian random variables with variance $\frac{N_o}{2E_{\tilde{u}}}$.

Since each of these detectors will have the same bit error probability, we study the real component. It has been specified that $|d_i| = 1$, so that $|\mathrm{Re}\,d_i| = |\mathrm{Im}\,d_i| = \frac{1}{\sqrt{2}}$. From the lectures, we know that the probability of bit error is given by

$$\mathrm{PBE} = \frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{0 - \frac{1}{\sqrt{2}}}{\sqrt{2}\sigma_{\mathrm{Re}\,N_z}}\right) = \frac{1}{2} - \frac{1}{2}\mathrm{erf}\left(\sqrt{\frac{E_{\tilde{u}}}{2N_o}}\right) = \frac{1}{2}\mathrm{erfc}\left(\sqrt{\frac{E_{\tilde{u}}}{2N_o}}\right)$$

There is a shortcut to this answer. Specifically, we know that the $\mathrm{PBE} = \frac{1}{2}\mathrm{erfc}\left(\sqrt{\frac{E_b}{N_o}}\right)$ where E_b is the average received bit energy. For this problem, the average symbol energy is $|d_i|^2E_{\tilde{u}} = E_{\tilde{u}}$ and there are two bits per symbol, giving the average bit energy $E_b = \frac{E_{\tilde{u}}}{2}$.

We can now use the fact that the bit error events are independent to write probability of word error as

$$\mathrm{PWE} = 1 - (1 - \mathrm{PBE})^2$$

3. Examining the expression for $E_{\tilde{u}}$, we see that

$$\begin{aligned}
E_{\tilde{u}} &= \int |\tilde{u}(t)|^2 dt \\
&= \sum_{k,l} h_k h_l^* \underbrace{\int u(t-\tau_k)u^*(t-\tau_l)dt}_{V_u(\tau_l - \tau_k)}
\end{aligned}$$

If $V_u(\tau_l - \tau_k) = 0$ for all $k \neq l$, then we have

$$E_{\tilde{u}} = \sum_k |h_k|^2 V_u(0) = \sum_k |h_k|^2 E_u$$

Since $u(t)$ has support on $[0, T_u]$, we cannot assume that $V_u(t) = 0$ for $t \in [-T_u, T_u]$. But if the difference between every pair of delays is greater than T_u, we satisfy the condition.

4. For particular $\{h_k\}$, we know that

$$\begin{aligned} Y_z(f) &= \int_{-\infty}^{\infty} Y_z(t) e^{-j2\pi f t} dt \\ &= D_z U(f) \sum_k h_k e^{-j2\pi f \tau_k} + W_z(f) \\ |Y_z(f)|^2 &= |D_z|^2 |U(f)|^2 \sum_{k,l} h_k h_l^* e^{-j2\pi f(\tau_k - \tau_l)} + |W_z(f)|^2 \end{aligned}$$

Taking an expectation conditioned on $\{h_k\}$ yields

$$\mathrm{E}_{\{h_k\}} |Y_z(f)|^2 = \left(\sum_{i=0}^{2^{K_b}-1} |d_i|^2 \pi_i \right) |U(f)|^2 \sum_{k,l} h_k h_l^* e^{-j2\pi f(\tau_k - \tau_l)} + N_o$$

and, finally, the full expectation is

$$\mathrm{E} |Y_z(f)|^2 = \left(\sum_{i=0}^{2^{K_b}-1} |d_i|^2 \pi_i \right) |U(f)|^2 \sum_k \sigma_k^2 + N_o$$