Professor Fitz
3/10/09

Problem fs.2

a) $R_{\tilde{N}} = E\left[\vec{\tilde{N}}\,\vec{\tilde{N}}^H\right]$

$$\left[R_{\tilde{N}}\right]_{ij} = E\left[\tilde{N}(i)\,\tilde{N}^*(j)\right]$$

$$= E\left[\int_0^{T_p+T_n} W_Z(t_1)\,\tilde{S}_i^*(t_1)\,dt_1 \int_0^{T_p+T_n} W_Z^*(t_2)\,S_j(t_2)\,dt_2\right]$$

$$= \int_0^{T_p+T_n}\int_0^{T_p+T_n} S_i^*(t_1)\,S_j(t_2)\;E\left[W_Z(t_1)\,W_Z^*(t_2)\right]dt_1 dt_2$$

$$= \int_0^{T_p+T_n}\int_0^{T_p+T_n} S_i^*(t_1)\,S_j(t_2)\,N_0\,\delta(t_1-t_2)\,dt_1\,dt_2$$

$$= N_0\int_0^{T_p+T_n} S_i^*(t)\,S_j(t)\,dt = N_0\,R_{\tilde{S}}(j,i)$$

$$R_{\tilde{N}} = N_0 E_b G$$

b) $\vec{N}_d = W^H\,\vec{\tilde{N}}$

$$R_{N_d} = E\left[\vec{N}_d\,\vec{N}_d^H\right] = E\left[W^H\vec{\tilde{N}}\,\vec{\tilde{N}}^H W\right]$$

$$= W^H R_{\tilde{N}}\,W = G^{-1} N_0 E_b\, G \left(G^{-1}\right)^H$$

$$= N_0 E_b\, G^{-1} \quad \text{since } \left(G^{-1}\right)^H = G^{-1}$$

Problem fs.2 (cont.)

c) $\hat{D}(k) = E_b\, P_z(k) + N_d(1)$

The decision is

$$\mathrm{Re}\left[\hat{D}(k)\right] \underset{\hat{I}(k)=1}{\overset{\hat{I}(k)=0}{\gtrless}} 0$$

If $I(k) = 0$

$$P_B(E, k \mid I(k)=0) = P\left(\hat{D}_I(k) < 0\right)$$

$$= P\left(E_b + N_{dI}(k) < 0\right)$$

$$= P\left(N_{dI}(k) < -E_b\right)$$

$$= \frac{1}{2}\,\mathrm{erfc}\left[\frac{E_b}{\sqrt{2}\sqrt{\mathrm{var}\left[N_{dI}(k)\right]}}\right]$$

$$\mathrm{var}\left[N_{dI}(k)\right] = \frac{1}{2}\,\mathrm{var}\left[N_d(k)\right] = \frac{1}{2}\, w_{KK} \cdot N_0 E_b$$

$$= \frac{1}{2}\,\mathrm{erfc}\left[\sqrt{\frac{E_b}{N_0 w_{KK}}}\right]$$

d) $$P_B(E) = \frac{1}{K_b} \sum_{k=1}^{K_b} \frac{1}{2}\,\mathrm{erfc}\left[\sqrt{\frac{E_b}{N_0 w_{KK}}}\right]$$

$$= \sum_k p_k\, g(w_{KK}) > g\left(\sum_k p_k\, w_{KK}\right)$$

Jensen's Inequality

Problem fs.2 (cont.)

where

$$A = \sum_K p_K w_{KK} = \frac{\text{trace}(W^H)}{K_b} = \sum_{K=1}^{K_b} \frac{\lambda_K^{(w)}}{K_b}$$

since

$$\text{trace}(W^H) = \left(\sum_{K=1}^{K_b} \lambda_K^{(w)}\right)$$

where $\lambda_K^{(w)}$ is the K^{th} eigenvalue of W^H

Note

1) $W^H = G^{-1} \implies \lambda_K^{(w)} = \left(\lambda_K^{(G)}\right)^{-1}$

2) The average energy constraint implies that

$$\sum_{K=1}^{K_b} \lambda_K^{(G)} = K_b \implies \sum_{K=1}^{K_b} \frac{\lambda_K^{(G)}}{K_b} = 1$$

$$A = \sum \frac{1}{K_b} \left(\lambda_K^{(G)}\right)^{-1} > \left(\sum \frac{1}{K_b} \lambda_K^{(G)}\right)^{-1} > 1$$

by Jensen's again

$$P_B(E) > \frac{1}{2} \operatorname{erfc}\left[\sqrt{\frac{E_b}{N_0} \cdot \frac{1}{\frac{1}{K_b}\sum_{K=1}^{K_b} W_{KK}}}\right] > \frac{1}{2} \operatorname{erfc}\left[\sqrt{\frac{E_b}{N_0}}\right]$$

e) Note if $W_{KK} = \infty$ is the only way that the error would not decrease exponentially

$W_{KK} = \infty \implies G$ is not a full rank

Problem fs.2 (cont.)

rank matrix and hence not invertible

Example

$$\vec{\tilde{Q}} = E_b \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \vec{D} + \vec{N}$$

Professor Fitz
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Problem fs.3

a) $R_{\tilde{N}} = E\left[\vec{\tilde{N}}\vec{\tilde{N}}^H\right]$

$\left[R_{\tilde{N}}\right]_{ij} = E\left[\tilde{N}(i)\tilde{N}^*(j)\right]$

$= E\left[\int_0^{T_p+T_n} W_z(t)\tilde{S}_i^*(t)\,dt \int_0^{T_p+T_n} W_z^*(\lambda)\tilde{S}_j(\lambda)\,d\lambda\right]$

$= \int_0^{T_p+T_n}\int_0^{T_p+T_n} \tilde{S}_i^*(t) S_j(\lambda)\; N_0\,\delta(t-\lambda)\,dt\,d\lambda$

$= N_0\, R_{\tilde{S}}(j, i)$

$R_{\tilde{N}} = N_0 E_b\, G$

b) $\vec{N}_m = W^H \vec{\tilde{N}}$

$R_{N_m} = W^H R_{\tilde{N}} W = \left(G + \frac{N_0}{E_b} I\right)^{-1} N_0 E_b G \left(G + \frac{N_0}{E_b} I\right)^{-1}$

c) $\vec{\hat{D}} = W^H G E_b \vec{D} + \vec{N}_m = E_b A \vec{D} + \vec{N}_m$

$\hat{D}(1) = E_b \sum_{\ell=1}^{4} a_{1\ell} D(\ell) + N_m(1)$

$$A = \begin{bmatrix} 0.9269 & -j0.0039 & -j0.0248 & -0.0213 \\ j0.0039 & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Problem fs.3 (cont.)

$$R_{N_m} = E_b N_o \; B$$

$$B = \begin{bmatrix} 0.6732 & \cdots & \\ \vdots & & \\ \vdots & & \\ \vdots & & 1.383 \end{bmatrix}$$

see Matlab

Decoder for first bit

$$\mathrm{Re}\left[\hat{D}(k)\right] \underset{\hat{I}(k)=1}{\overset{\hat{I}(k)=0}{\gtrless}} 0$$

Assume $I(k)=0$ and note only one bit effects the real part, $I(4)$,

$$P_B(E, 1) = P\left(0.9269\,E_b - 0.0213\,E_b D_Z(4) + N_{mI}(1) < 0\right)$$

$$= \frac{1}{2} P\left(0.9056\,E_b + N_{mI}(1) < 0\right) +$$

$$\frac{1}{2} P\left(0.9482\,E_b + N_{mI}(1) < 0\right)$$

$$P\left(0.9056\,E_b + N_{mI}(1) < 0\right) = \frac{1}{2}\,\mathrm{erfc}\left[\frac{0.9056\,E_b}{\sqrt{N_0 E_b (0.6732)}}\right]$$

$$P\left(0.9482\,E_b + N_{mI}(1) < 0\right) = \frac{1}{2}\,\mathrm{erfc}\left[\frac{0.9482\,E_b}{\sqrt{N_0 E_b (0.6732)}}\right]$$

```
% Digital Communication Theory
%
% MMSE Detector analysis
%
% Author: M. Fitz
% Last Revision: 3/10/04
%
A=[0.9269 -j*0.0039 -j*0.0248 0.0123;
   j*0.0039  0.8539 0.0567 -j*0.1094;
   j*0.0248 0.0567 0.8208 j*0.0807;
   -0.012 j*0.1094  -j*0.0807   0.8051]
W_herm=[0.7305 j*0.03852 j*0.2482 0.1231;
        -j*0.0385 1.4609 -0.5669 j*1.0939;
        -j*0.2482 -0.5669 1.7919 -j*0.8072;
        0.1231 -j*1.0939 j*0.8072 1.9492]
B=A*W_herm'
xlabel('E_b/N_0, dB')
ylabel('P_{WUB}(E)')
legend('Frequency Flat','Channel 1','Channel 2')
```