Problem MC.1

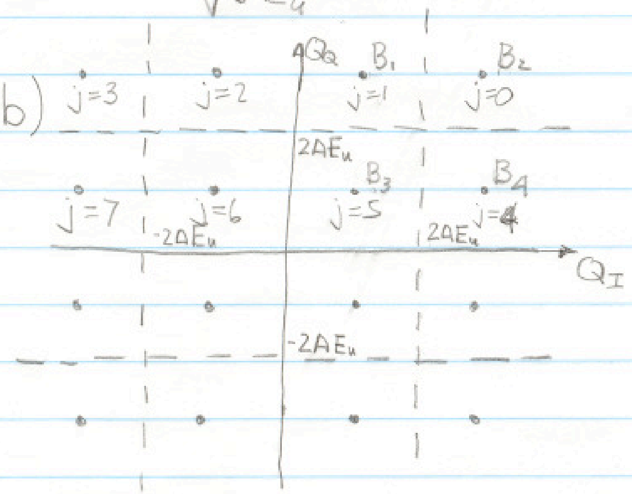
a) $D_i = D_{Ii} + j D_{Qi}$

$$E_b = \frac{\frac{1}{16}\sum_{i=0}^{15} |D_i|^2 E_u}{4} = \frac{E_u \cdot (2+10+10+18)A^2}{16}$$

$$= E_u \cdot \frac{5}{2} A^2$$

$$A = \sqrt{\frac{2E_b}{5E_u}}$$

b)



The decision regions for 16QAM is show in the figure.

$$P_W(E) = \frac{1}{16}\sum_{i=0}^{15} P_W(E|D_i)$$

c)

$P_W(E|D_i)$ has 4 four forms

$\underline{D_i = B_1}$

$$P_W(E/B_1) = P\left(\{Q_I \le 0\} \cup \{2AE_u \le Q_I\} \cup \{Q_Q \le 2AE_u\} \middle| B_1\right)$$

note: $Q = (A + j3A)E_u + N_Z$ for B_1
$var[N_Z] = E_u N_0$

$$P_W(E|B_1) = 1 - P\left(\{0 \le Q_I \le 2AE_u\} \cap \{2AE_u \le Q_Q\} \middle| B_1\right)$$

Problem mc.1 (cont.)

N_I is independent of N_Q so

$$P_w(E/B_1) = 1 - P(\{0 \le Q_I \le 2AE_u\})P(\{2AE_u \le Q_Q\})$$

$$= 1 - \operatorname{erf}\left(\frac{AE_u}{\sqrt{E_u N_0}}\right)\left(\frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{AE_u}{\sqrt{E_u N_0}}\right)\right)$$

$$= 1 - \operatorname{erf}\left(\sqrt{\frac{2E_b}{5N_0}}\right)\left(\frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\sqrt{\frac{2E_b}{5N_0}}\right)\right)$$

$\underline{D_i = B_2}$

$$P_w(E/B_2) = 1 - \left(\frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\sqrt{\frac{2E_b}{5N_0}}\right)\right)\left(\frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\sqrt{\frac{2E_b}{5N_0}}\right)\right)$$

$\underline{D_i = B_3}$

$$P_w(E/B_3) = 1 - \operatorname{erf}\left(\sqrt{\frac{2E_b}{5N_0}}\right)\operatorname{erf}\left(\sqrt{\frac{2E_b}{5N_0}}\right)$$

$\underline{D_i = B_4}$

$$P_w(E/B_4) = 1 - \left(\frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\sqrt{\frac{2E_b}{5N_0}}\right)\right)\operatorname{erf}\left(\sqrt{\frac{2E_b}{5N_0}}\right)$$

since the condition error rate is different for at least some of the constellation points then the signal set is not geometrically uniform

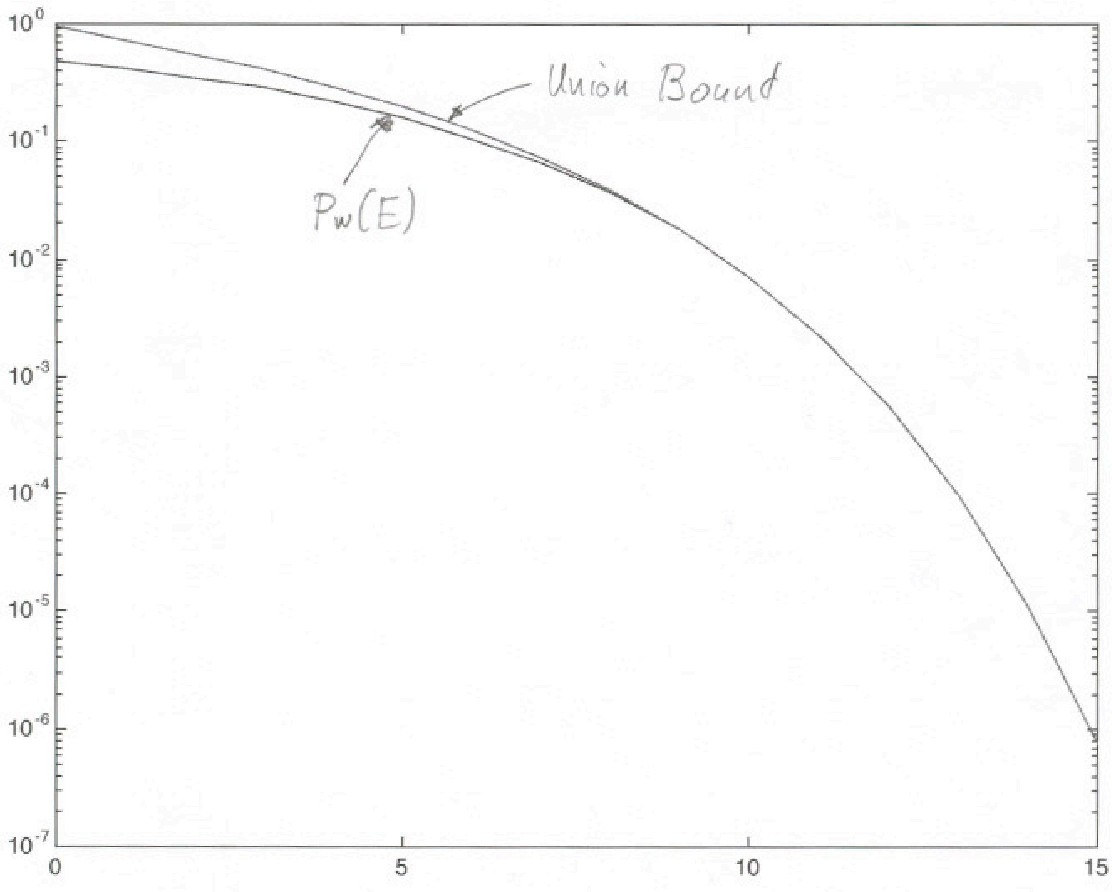
d) $P_w(E) = \frac{1}{4}P_w(E/B_1) + \frac{1}{4}P_w(E/B_2) + \frac{1}{4}P_w(E/B_3) + \frac{1}{4}P_w(E/B_4)$

see m-file and plot

e) $$P_w(E) \le \sum_{j=1}^{16} \sum_{\substack{i=1 \\ i \ne j}}^{16} \frac{1}{2}\operatorname{erfc}\left[\sqrt{\frac{|D_i - D_j|^2 E_u}{N_0}}\right]$$

This is best automated as their are 16×15 terms (see m-file)

```
% EE894 Digital Communication Theory
% Chapter 7 Problem 1
%
% Author: M. Fitz
% Last Revision: 6/5/00
%
%  Part b)
% Assume N_0=1
%
ebdb=(0:15)';
eb=10.^(ebdb/10);
p_wb1=1-erf(sqrt(2*eb/5)).*(0.5+0.5*erf(sqrt(2*eb/5)));
p_wb2=1-(0.5+0.5*erf(sqrt(2*eb/5))).*(0.5+0.5*erf(sqrt(2*eb/5)));
p_wb3=1-erf(sqrt(2*eb/5)).*erf(sqrt(2*eb/5));
p_wb4=1-erf(sqrt(2*eb/5)).*(0.5+0.5*erf(sqrt(2*eb/5)));
pwe=(p_wb1+p_wb2+p_wb3+p_wb4)/4;
semilogy(ebdb,pwe)
%  Part c)
% Assume N_0=1
d_i=[-3+j*3 -1+j*3 1+j*3 3+j*3 -3+j*1 -1+j*1 1+j*1 3+j*1 -3+j*-1 -1+j*-1 1+j*-1 3+j*-1 -3+j*-3
-1+j*-3 1+j*-3 3+j*-3];
pub=0;
for jj=1:16
  for kk=1:16
    if(jj~=kk)
      delta_E=(d_i(jj)-d_i(kk))*conj(d_i(jj)-d_i(kk))*2/5*eb;
       pub=pub+0.5*erfc(sqrt(delta_E/4));
    end
  end
  %[jj pub]
end
p  (pub/16)
semilogy(ebdb,pwe,ebdb,pub)
```



Union Bound
$P_w(E)$
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
0
5
10
15

Problem MC.1 (Cont.)

f) Again for reducing down the union bound there are 4 unique constellation element types to be considered.

$$B = \{B_1, B_2, B_3, B_4\}$$

The solution will consider B_1 in detail as the results for the other three points are similar

For $B_1 \Leftrightarrow j = j_1$

$$P_W(E|B_1) = P\left(\bigcup_{\substack{i=0 \\ i \neq 1}}^{15} \left\{ |Q - D_i A E_u|^2 < |Q - D_{j_1} A E_u|^2 \right\}\right)$$

$$= P\left(\bigcup_{\substack{i=0 \\ i \neq 1}}^{15} \{T_i > T_1\}\right) =$$

Note
$$\bigcup_{\substack{i=0 \\ i \neq 1}}^{15} \{T_i > T_1\} = \{T_2 > T_1\} \cup \{T_5 > T_1\} \cup \{T_0 > T_1\}$$

$$P\{T_2 > T_1\} = \frac{1}{2}\operatorname{erfc}\left[\sqrt{\frac{2E_b}{5N_0}}\right] = P\{T_5 > T_1\} = P\{T_0 > T_1\}$$

The union bound for B_1 becomes

$$P_W(E|B_1) = \frac{3}{2}\operatorname{erfc}\left[\sqrt{\frac{2E_b}{5N_0}}\right]$$

Problem mc.1 (cont.)

It is interesting to compare this union bound with the exact $P_w(E|B_1)$, i.e.,

$$P_w(E|B_1) = 1 - \left(1 - \text{erfc}\left(\sqrt{\frac{2E_b}{5N_0}}\right)\right)\left(1 - \frac{1}{2}\text{erfc}\left[\sqrt{\frac{2E_b}{5N_0}}\right]\right)$$

$$= \frac{3}{2}\text{erfc}\left[\sqrt{\frac{2E_b}{5N_0}}\right] - \frac{1}{2}\left(\text{erfc}\left[\sqrt{\frac{2E_b}{5N_0}}\right]\right)^2$$

The dominating term is the same but the overlapped region get subtracted out in the true $P_w(E)$.