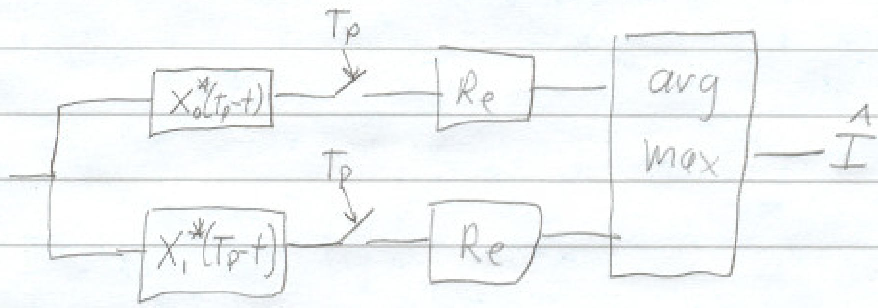
Mike Fitz
1/27/03

Problem sb.10

a)



b) $$P_B(E) = \frac{1}{2}\operatorname{erfc}\left[\sqrt{\frac{\Delta_E(1,0)}{4N_0}}\right]$$

$$\Delta_E(1,0) = \int_0^{T_p} |x_1(t) - x_0(t)|^2 dt$$

$$= 2E_b - \frac{2E_b}{T_p}\int_0^{T_p} \mathrm{Re}\left[\exp\left[j\frac{\pi t}{2T_p} - j\theta\right]\right] dt$$

$$= 2E_b - \frac{2E_b}{T_p}\mathrm{Re}\left[e^{-j\theta}\int_0^{T_p}\exp\left[\frac{j\pi t}{2T_p}\right]dt\right]$$

$$= 2E_b - 2E_b\,\mathrm{Re}\left[e^{-j\theta}\left(\frac{\sin(\frac{\pi}{2})}{\frac{\pi}{2}} - j\frac{\cos(\frac{\pi}{2}) - 1}{\frac{\pi}{2}}\right)\right]$$

$$= 2E_b - \frac{4E_b}{\pi}\mathrm{Re}\left[e^{-j\theta}(1+j)\right]$$

maximized for $\theta = -\frac{3\pi}{4}$

$$\Delta_E(1,0) = 2E_b\left(1 + \frac{4\sqrt{2}}{\pi}\right) = 3.8E_b < 4E_b \text{ for BPSK}$$

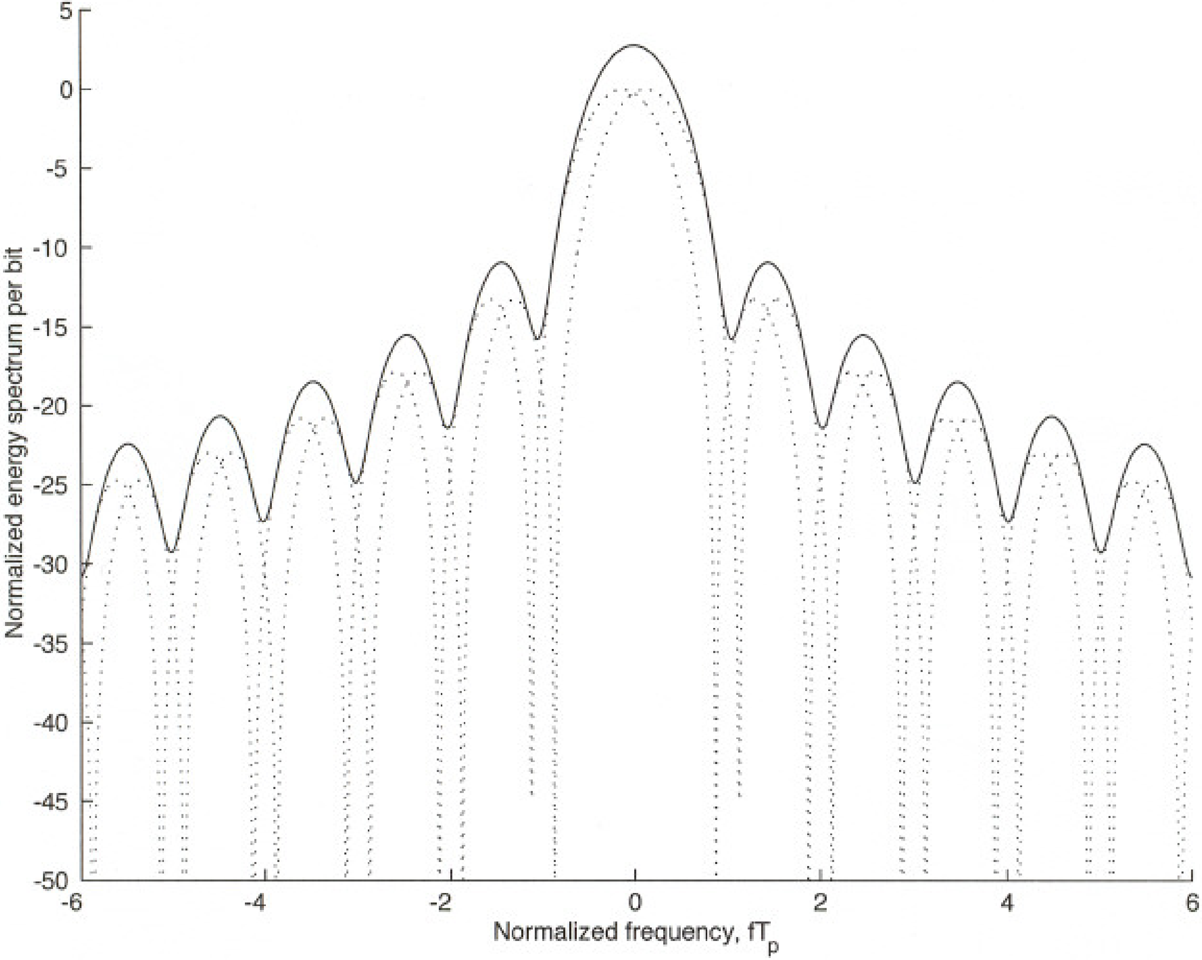
c) $$G_{x_0}(f) = E_b T_p\left(\frac{\sin(\pi f T_p - \pi T_p/8)}{\pi f T_p - \pi T_p/8}\right)^2$$

Problem s6.10 (cont.)

$$G_{x_1}(f) = E_b T_p \left(\frac{\sin(\pi f T_p + \pi T_p/8)}{\pi f T_p + \pi T_p/8} \right)^2$$

$$D_{x_2}(f) = \frac{1}{2}\left(G_{x_0}(f) + G_{x_1}(f) \right)$$

see attached plot. (using bfskspec.m from webpage)



5
0
-5
-10
-15
-20
-25
-30
-35
-40
-45
-50
-6
-4
-2
0
2
4
6
Normalized energy spectrum per bit
Normalized frequency, fT_p

Prof. Fitz

Problem 5b.13

a) This modulation has the form $x_i(t) = d_i u(t)$

$$Q = \int Y_z(t) u^*(t)\,dt = \int_0^{T_p/2} Y_z(t) \exp\left[-j2\pi f_1 t\right] dt + \int_{T_p/2}^{T_p} Y_z(t) \exp\left[-j2\pi f_2 t\right] dt$$

$$E_i = |d_i|^2 E_u = |d_i|^2 E_b$$

$$T_0 = \mathrm{Re}\left[d_0^* Q\right] - \frac{|d_0|^2 E_b}{2}$$

$$T_1 = \mathrm{Re}\left[d_1^* Q\right] - \frac{|d_1|^2 E_b}{2}$$

b) $$P_B(E) = \frac{1}{2}\,\mathrm{erfc}\left[\sqrt{\frac{\Delta_E(1,0)}{4N_0}}\right]$$

$$\Delta_E(1,0) = |d_0 - d_1|^2 E_b$$

c) $d_0 = 1$, $d_1 = -1$ $\Rightarrow \Delta_E(1,0) = 4E_b$

d) Only point of possible discontinuity is at $t = T_p/2$

$$2\pi f_1 T_p = 2\pi f_2 T_p$$

$$2\pi f_2 T_p = 2\pi f_1 T_p + 2\pi n \quad \text{where } n \text{ is an integer}$$

$$f_2 = f_1 + \frac{n}{T_p}$$

8

Problem sb.13 (cont.)

$$G_{x_0}(f) = |X_0(f)|^2 = |d_0|^2 |u(f)|^2$$

$$u(t) = u_1(t) + u_2(t)$$

$$u_1(t) = \begin{cases} \sqrt{\frac{E_b}{T_p}} \exp\left[j 2\pi f_1 t\right] & 0 \le t \le \frac{T_p}{2} \\ 0 & \text{elsewhere} \end{cases}$$

$$u_2(t) = \begin{cases} \sqrt{\frac{E_b}{T_p}} \exp\left[j 2\pi f_2 t\right] & \frac{T_p}{2} \le t \le T_p \\ 0 & \text{elsewhere} \end{cases}$$

$$U_1(f) = \sqrt{E_b T_p} \exp\left[-j \frac{\pi f T_p}{2}\right] \operatorname{sinc}\left(\frac{(f-f_1)T_p}{2}\right)$$

$$U_2(f) = \sqrt{E_b T_p} \exp\left[-j \frac{3\pi f T_p}{2}\right] \operatorname{sinc}\left(\frac{(f-f_2)T_p}{2}\right)$$

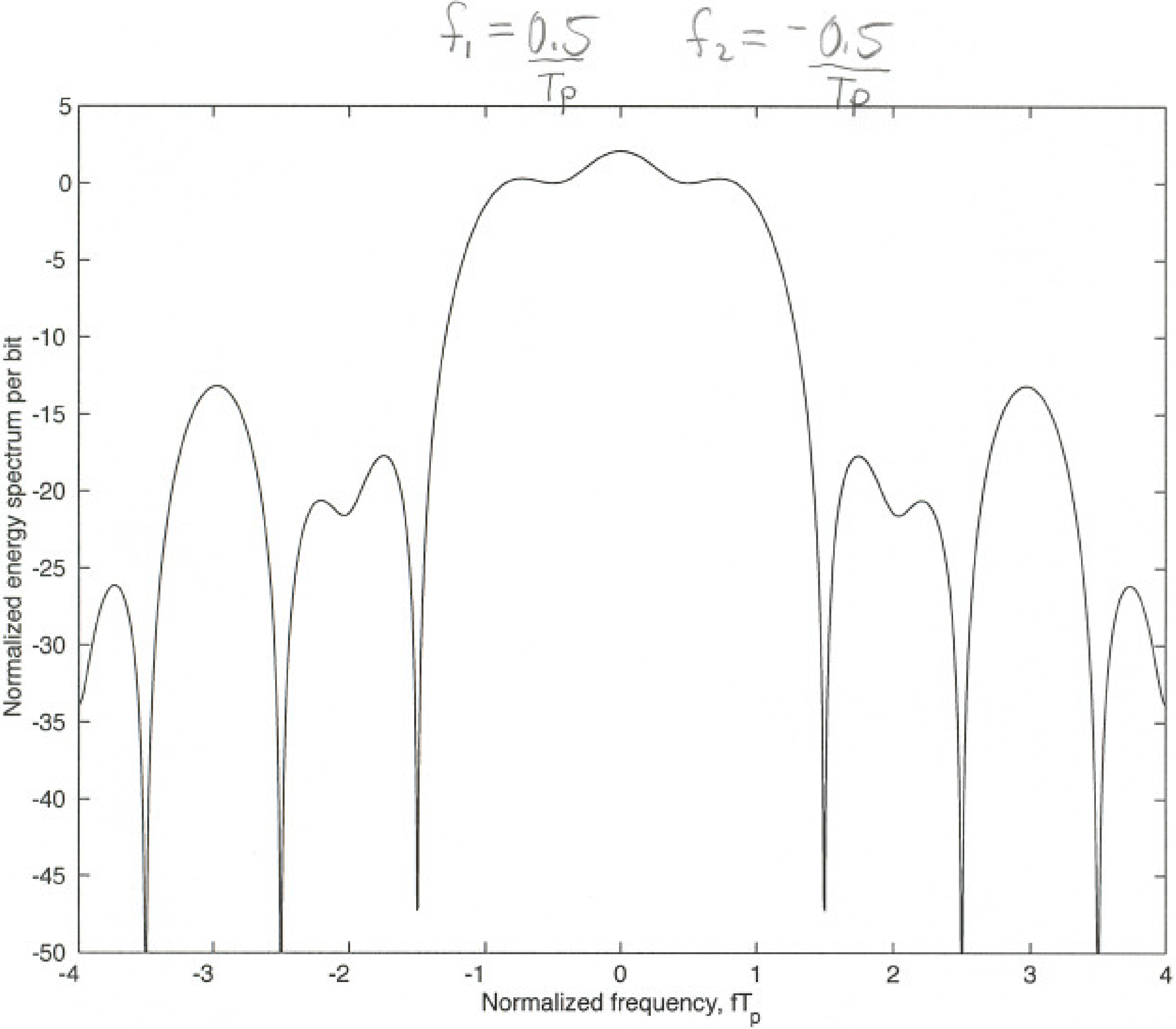
$$G_{x_0}(f) = |d_0|^2 |U_1(f) + U_2(f)|^2$$

$$G_{x_1}(f) = |d_1|^2 |U_1(f) + U_2(f)|^2$$

$$D_{x_z}(f) = \frac{|d_0|^2 + |d_1|^2}{2} |U_1(f) + U_2(f)|^2$$

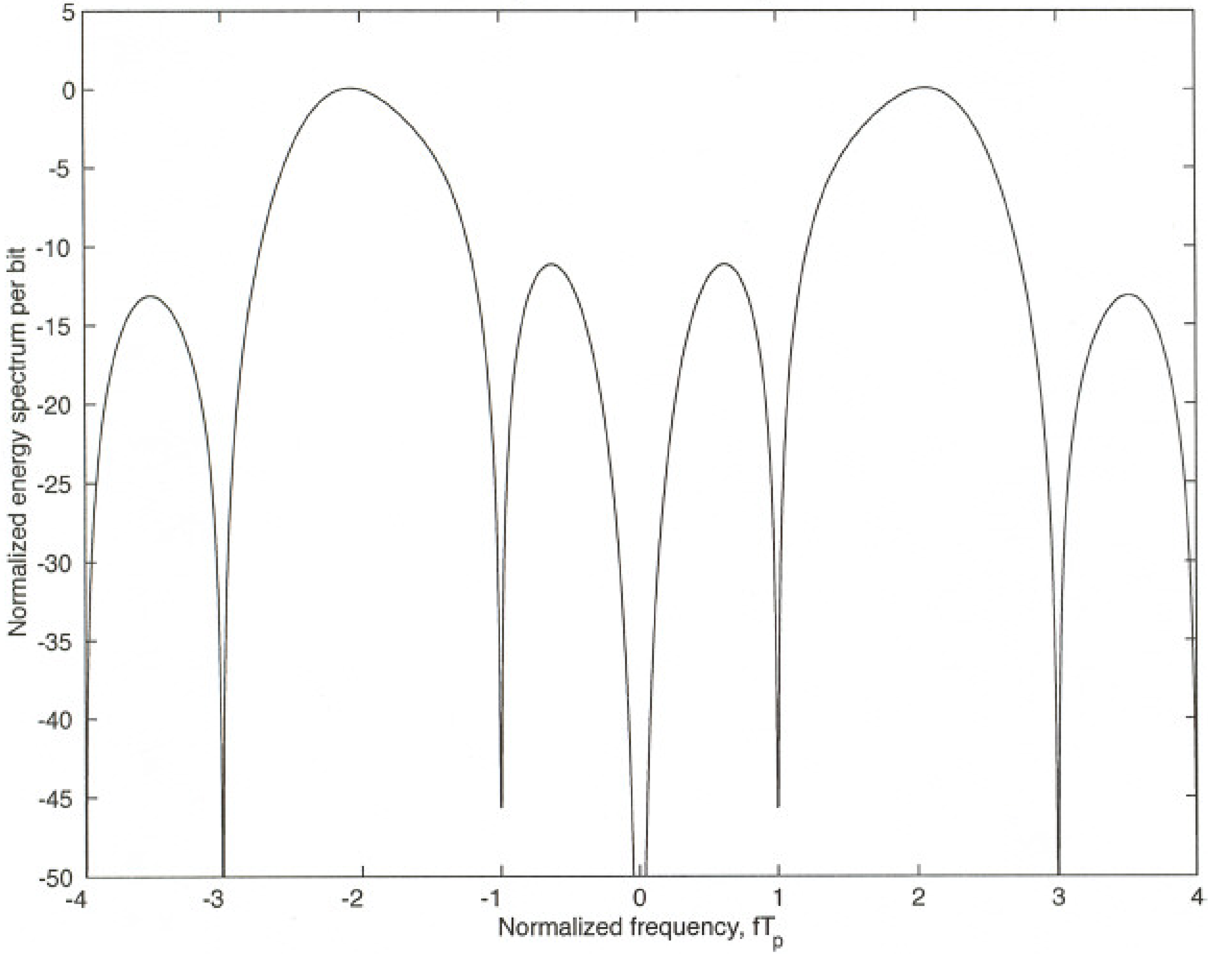
see matlab code and plot

```
%
% Generating the spectrum of OFDM
% Author: M. Fitz
% Last modified: 1/28/04
%
close all
clear all
numpts=2048;
f_1=2;
f_2=-2;
freq=linspace(-4,4, numpts+1);
g_pr=zeros(1,numpts+1);
dum1=exp(-j*pi*freq/2).*sinc(freq-f_1*ones(1,numpts+1));
dum2=exp(-j*3*pi*freq/2).*sinc(freq-f_2*ones(1,numpts+1));
g_pr=(dum1+dum2).*conj(dum1+dum2);
g_prdb=10*log10(g_pr);
figure(1)
plot(freq,g_prdb)
axis([-4 4 -50 5])
xlabel('Normalized frequency, fT_p')
ylabel('Normalized energy spectrum per bit')
hold off
```



$f_1 = \frac{0.5}{T_p}$
$f_2 = -\frac{0.5}{T_p}$
5
0
-5
-10
-15
-20
-25
-30
-35
-40
-45
-50
-4
-3
-2
-1
0
1
2
3
4
Normalized energy spectrum per bit
Normalized frequency, fT_p

$f_1 = \frac{2}{T_p} \qquad f_2 = \frac{-2}{T_p}$



M. Fitz 5/3/00

Problem mb. 1

$$P_w(E) = P\left(\bigcup_{\substack{i=0 \\ j \neq i}}^{M-1} \{T_{i|j} > T_{j|j}\} \right)$$ due to equal energy and orthogonality

$$T_{i|j} = \frac{-E_s}{2} + N_I^{(i)} \qquad \text{var}\left[N_I^{(i)}\right] = \frac{E_s N_0}{2}$$

$$T_{j|j} = \frac{E_s}{2} + N_I^{(j)} \qquad \text{Re}[\rho_{ij}] = 0 \Rightarrow \text{all } T_{i|j} \text{ are independent R.V.}$$

$$P_w(E) = \int_{-\infty}^{\infty} P\left(\max_{\substack{i=0, M-1 \\ j \neq i}} T_{i|j} > t \right) f_{T_{j|j}}(t)\, dt$$

⇒ here we have conditioned on $T_{j|j} = t$ and used total probability

$$= \int_{-\infty}^{\infty} \left(1 - F_Z(t)\right) f_{T_{j|j}}(t)\, dt$$

where $Z = \max\limits_{\substack{i=0, M-1 \\ j \neq i}} T_{i|j}$

$$F_Z(t) = \prod_{\substack{i=0 \\ j \neq i}}^{M-1} F_{T_{i|j}}(t) = \left(F_{T_{i|j}}(t)\right)^{M-1}$$ Papoulis Section 7-1

$$F_{T_{i|j}}(t) = \frac{1}{2} + \frac{1}{2}\,\text{erf}\left(\frac{t + \frac{E_s}{2}}{\sqrt{E_s N_0}} \right)$$

$$f_{T_{j|j}}(t) = \frac{1}{\sqrt{\pi E_s N_0}} \exp\left[- \frac{\left(t - \frac{E_s}{2}\right)^2}{E_s N_0} \right]$$

See attached Matlab plot and Figure 6.4 in the notes.

```
% EE894 Digital Communication Theory
% Chapter 6 Problem 1
%
% Author: M. Fitz
% Last Revision: 4/13/00
%
% Assume N_0=1
%
p_wea=[];
p_uba=[];
fnorma=[];
k_b=4;
msig=2^k_b;
for E_bdb=0:10
E_b=10^(E_bdb/10);
E_s=k_b*E_b;
stddev=sqrt(E_s/2);
%
% Considering the true MF out for -numstd stddev
% numpts= number of points considered in the integration
% h=trapezoidal integration step size
%
numstd=7.0;
numpts=5001;
h=2*numstd*stddev/(numpts-1);
%
% Integrating over +- numstd*sigma
%
arg1=linspace(E_s/2-numstd*stddev,E_s/2+numstd*stddev,numpts);
p_wccond=(1-0.5*erfc((arg1+E_s/2)/sqrt(E_s))).^(msig-1);
fnormal=exp(-(arg1-E_s/2).^2/E_s)/sqrt(pi*E_s);
%
% Trapezoidal integration
% Subtracting off half of the end points
%
p_wc1a=(1-0.5*erfc((E_s/2-numstd*stddev)/sqrt(E_s))).^(msig-1);
fnormal1a=exp(-(-numstd*stddev).^2/(E_s))/sqrt(pi*E_s);
p_wc1=-0.5*p_wc1a*fnormal1a;
p_wc1b=(1-0.5*erfc((E_s/2+numstd*stddev)/sqrt(E_s))).^(msig-1);
fnormal1b=exp(-(numstd*stddev).^2/(E_s))/sqrt(pi*E_s);
p_wc1=p_wc1-0.5*p_wc1b*fnormal1b;
fnorm1=-0.5*(fnormal1a+fnormal1b);
%p_wc1=-0.5*exp(-(E_s-numstd*stddev-E_s).^2/(2*E_s))/sqrt(2*pi*E_s);
%p_wc1=p_wc1-0.5*exp(-(E_s+numstd*stddev-E_s).^2/(2*E_s));
p_wc=(p_wc1+sum(p_wccond.*fnormal))*h;
fnorm=(fnorm1+sum(fnormal))*h
% weighting the other half of the density with minimum
p_we=1-p_wc
p_ub=(msig-1)*0.5*erfc(sqrt(E_s/2))
p_be=msig/2/(msig-1)*p_we
p_wea=[p_wea p_we]
p_uba=[p_uba p_ub]
fnorma=[fnorma fnorm]
end
p_wea'
p_uba'
%fnorma'
```

Problem mb.7

a) $\vec{I} = \arg\max_{i=0,3} T_i$

$$T_i = \mathrm{Re}\left[\int_0^{T_p} Y_z(t)\, x_i^*(t)\, dt\right]$$

b) $j=0$ $\quad \Delta_E(1,0) = 2$ $\quad \Delta_E(2,0) = 4$ $\quad \Delta_E(3,0) = 2$

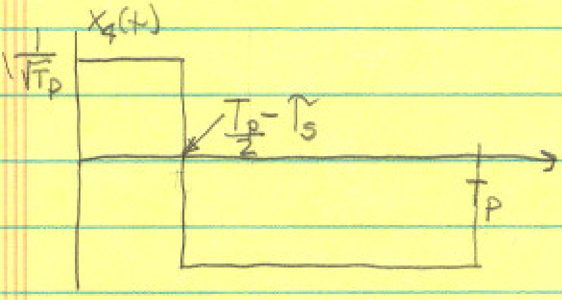
The other symbols have same distance spectrum

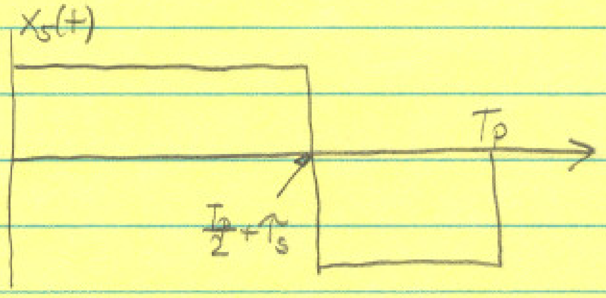
$$P_W(E) < \mathrm{erfc}\left[\sqrt{\frac{1}{2N_0}}\right] + \frac{1}{2}\mathrm{erfc}\left[\sqrt{\frac{1}{N_0}}\right]$$

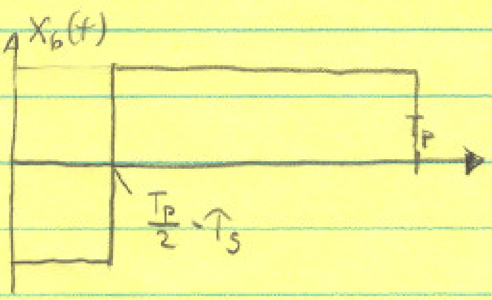
c) See plot

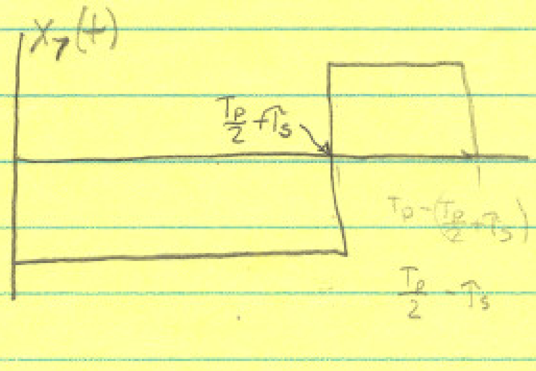
$B_3 = \frac{1.8}{T_p}$ $\qquad W_b = \frac{2}{T_p}$ $\qquad \eta_B = \frac{10}{9} = 1.11$

d) 4 waveforms from Figure 6.9 plus



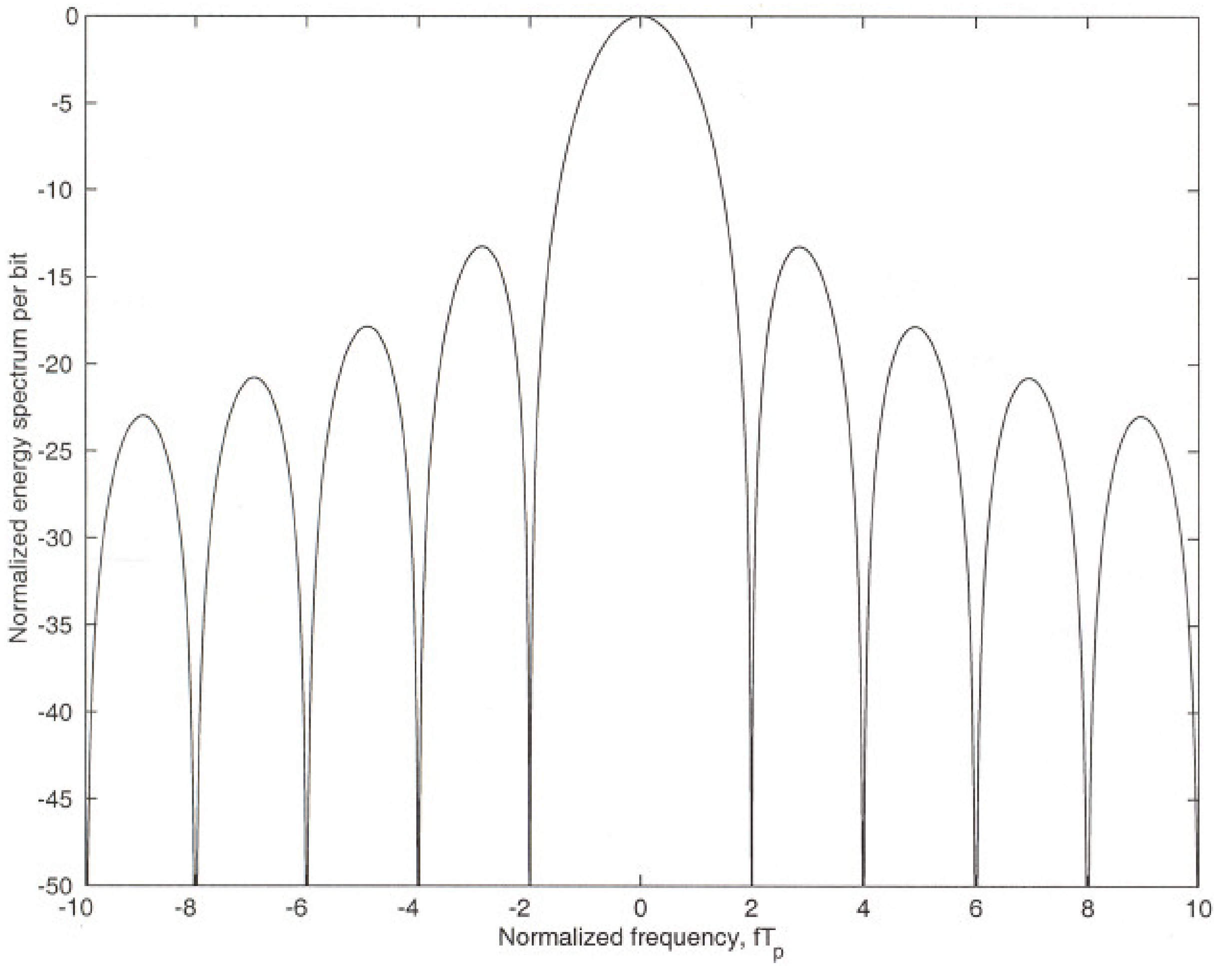






mb.7

part c)



0
-5
-10
-15
-20
-25
-30
-35
-40
-45
-50
-10
-8
-6
-4
-2
0
2
4
6
8
10
Normalized energy spectrum per bit
Normalized frequency, fT_p

Problem m6.7

e) Euclidean distance spectrum

$j=0$

$$\Delta_E(1,0)=2$$
$$\Delta_E(2,0)=4$$
$$\Delta_E(3,0)=2$$
$$\Delta_E(4,0)=2+4\frac{\tau_s}{T_p}$$
$$\Delta_E(5,0)=2-4\frac{\tau_s}{T_p}$$
$$\Delta_E(6,0)=2-4\frac{\tau_s}{T_p}$$
$$\Delta_E(7,0)=2+4\frac{\tau_s}{T_p}$$

$j=1$

$$\Delta_E(0,1)=2$$
$$\Delta_E(2,1)=2$$
$$\Delta_E(3,1)=4$$
$$\Delta_E(4,1)=4\frac{\tau_s}{T_p}$$
$$\Delta_E(5,1)=4\frac{\tau_s}{T_p}$$
$$\Delta_E(6,1)=2+4\frac{\tau_s}{T_p}$$
$$\Delta_E(7,1)=4-4\frac{\tau_s}{T_p}$$

$j=4$

$$\Delta_E(0,4)=2+4\frac{\tau_s}{T_p}$$
$$\Delta_E(1,4)=4\frac{\tau_s}{T_p}$$
$$\Delta_E(2,4)=2-4\frac{\tau_s}{T_p}$$
$$\Delta_E(3,4)=4-4\frac{\tau_s}{T_p}$$
$$\Delta_E(5,4)=8\frac{\tau_s}{T_p}$$
$$\Delta_E(6,4)=4$$
$$\Delta_E(7,4)=8\frac{\tau_s}{T_p}$$

other cases are similar.

Problem mb.7

The minimum distance will be maximize when

$$4\frac{\tau_s}{T_p} = 2 - 4\frac{\tau_s}{T_p}$$

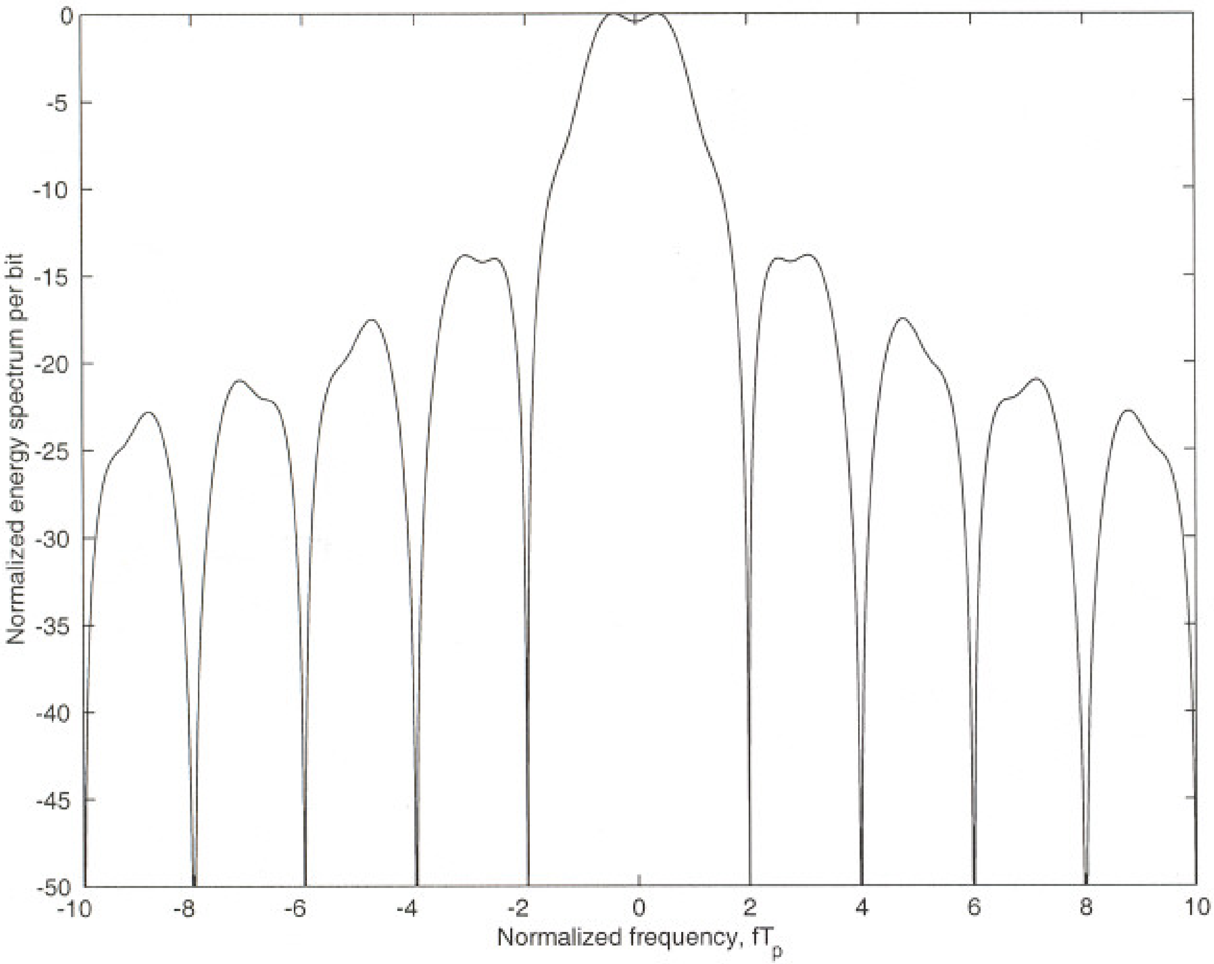
$$\frac{\tau_s}{T_p} = \frac{1}{4}$$

$$P_W(E) \approx A\,\mathrm{erfc}\left[\sqrt{\frac{1}{4N_0}}\right]$$

Performance is about 3dB worse

f) $B_3 \approx \frac{1.8}{T_p}$ $\quad W_b = \frac{3}{T_p}$ $\quad \eta_B = 1.67$

⇒ VPSK gives an improvement in spectral efficiency but at a significant loss in performance part f)

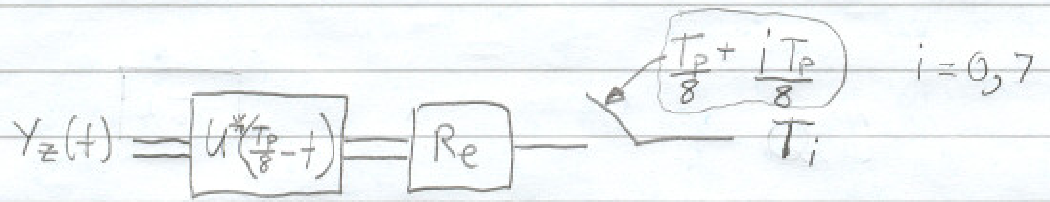


Normalized energy spectrum per bit
0
-5
-10
-15
-20
-25
-30
-35
-40
-45
-50
-10
-8
-6
-4
-2
0
2
4
6
8
10
Normalized frequency, fT_p

Mike Fitz
1/28/04

Problem mb.13

a)



$$T_i = \mathrm{Re}\left[\int_{\frac{iT_p}{8}}^{\frac{(i+1)T_p}{8}} Y_z(t)\,dt\right] \Rightarrow \text{only one filter sampled 8 times}$$

note PPM is an equal energy modulation scheme

$$\hat{I} = \arg\max_{i=0,7} T_i$$

b) Note

$$\Delta_E(i,j) = 6E_b \qquad i \neq j \qquad 8PPM$$

$$\Delta_E(i,j) = 6E_b \qquad i \neq j \qquad 8FSK$$

Hence the union bounds are identical

c) $X_i(f) = \exp\left[j\frac{(2i+1)}{8}\pi f T_p\right] \mathrm{sinc}\left(\frac{fT_p}{8}\right) \qquad i=0,7$

$$G_{X_i}(f) = \mathrm{sinc}^2\left(\frac{fT_p}{8}\right)$$

$$D_{x_z}(f) = \frac{\mathrm{sinc}^2\left(\frac{fT_p}{8}\right)}{3}$$

see attach Matlab code and plot

Problem mb.13 (cont.)

d) PPM has a worse behaved spectrum
PPM has a higher peak to average power ratio
Both has same performance
Complexity of PPM slightly smaller

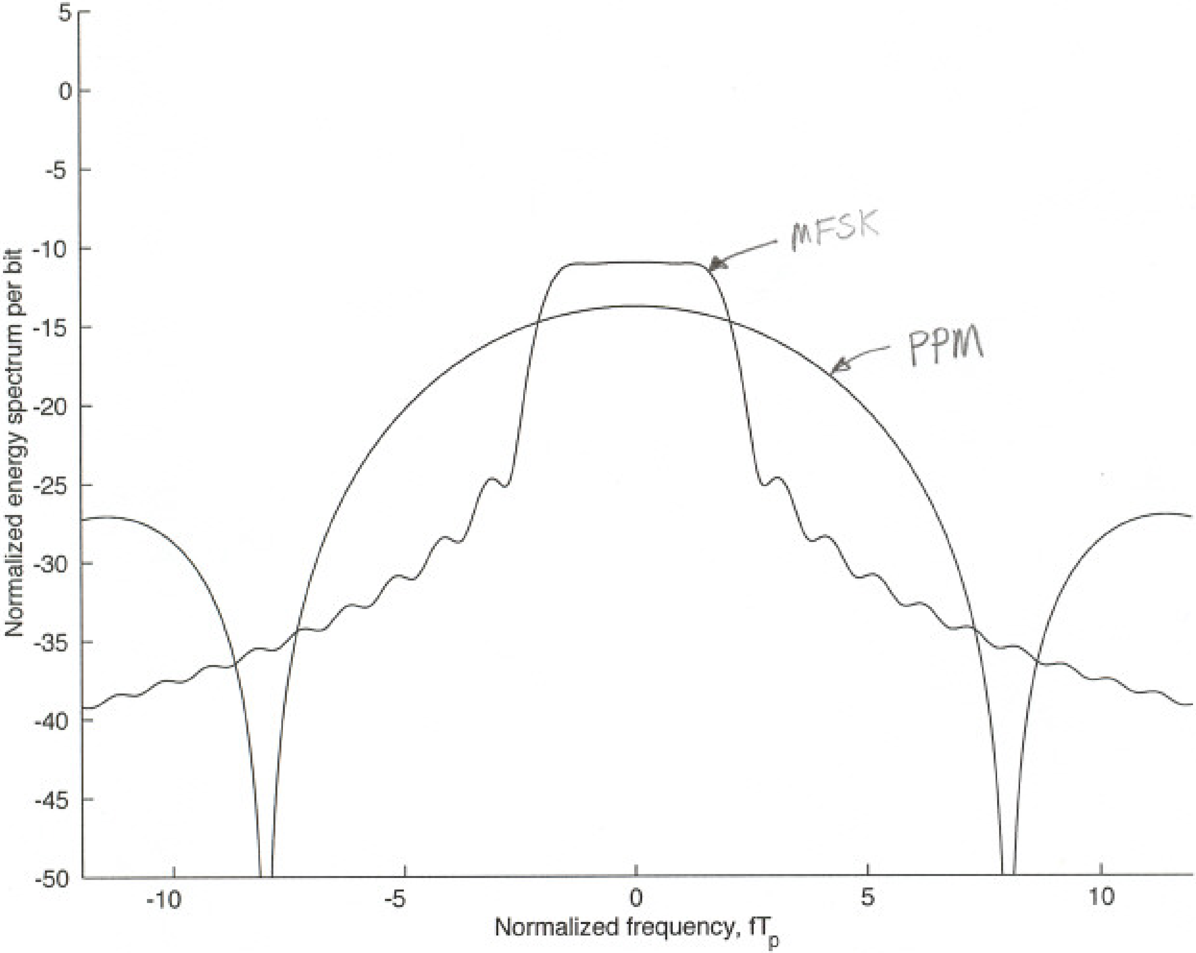
e)

I(1)	I(2)	I(3)	i	m	n
0	0	0	0	0	0
0	0	1	1	0	1
0	1	0	2	0	2
0	1	1	3	0	3
1	0	0	4	1	0
1	0	1	5	1	1
1	1	0	6	1	2
1	1	1	7	1	3

$$L_0(1) = \exp\left[\frac{2T_0}{N_0}\right] + \exp\left[\frac{2T_1}{N_0}\right] + \exp\left[\frac{2T_2}{N_0}\right] + \exp\left[\frac{2T_3}{N_0}\right]$$

$$L_1(1) = \exp\left[\frac{2T_4}{N_0}\right] + \exp\left[\frac{2T_5}{N_0}\right] + \exp\left[\frac{2T_6}{N_0}\right] + \exp\left[\frac{2T_7}{N_0}\right]$$

```
%
% Generating the spectrum of MFSK
% Author: M. Fitz
% Last modified: 1/28/04
%
close all
clear all
numpts=2048;
K_b=3;
M=2^K_b;
f_d=0.25;
freq=linspace(-8-M/2,8+M/2, numpts+1);
g_pr=zeros(1,numpts+1);
figure(1)
hold on
for kk=0:M-1
    dum1=sinc(freq-(2*kk-M+1)*f_d*ones(1,numpts+1)).^2;
    dumdb=10*log10(dum1);
    %plot(freq,dumdb,'r:')
    g_pr=g_pr+dum1;
end
g_fskdb=10*log10(g_pr/(M*K_b));
g_ppm=sinc(freq/M).^2/M;
g_ppmdb=10*log10(g_ppm/(K_b));
plot(freq,g_fskdb,freq,g_ppmdb)
axis([-8-M/2 8+M/2 -50 5])
xlabel('Normalized frequency, fT_p')
ylabel('Normalized energy spectrum per bit')
hold off
```



MFSK
PPM
Normalized energy spectrum per bit
Normalized frequency, fT_p
5
0
-5
-10
-15
-20
-25
-30
-35
-40
-45
-50
-10
-5
0
5
10