Communication of Multiple Bits: [Ch. 8]

Problem Setup:

- $\underline{I} \in \{0, 1, 2, \ldots, M-1\}, \quad M = 2^{K_b}$
- $\pi_i \triangleq P(\underline{I} = i)$
- $I = i \quad \Rightarrow \quad X_z(t) = x_i(t)$ with time support on $[0, T_p]$
- $Y_z(t) = X_z(t) + W_z(t)$ for CWGN with $S_{W_z}(f) = N_o$

Goals:

1. Optimum word demodulation
2. Optimum bit demodulation

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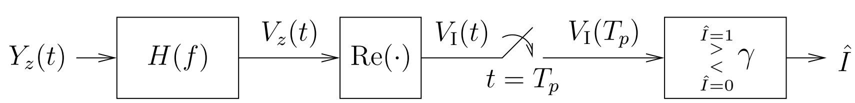
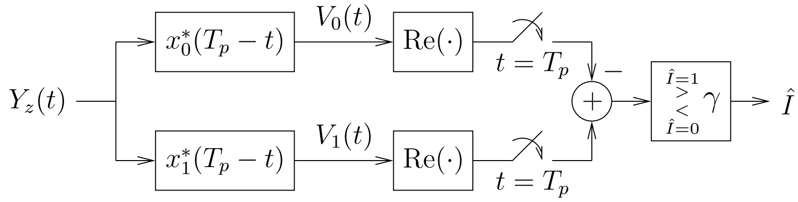
Recall binary demodulation



With equal priors ($\pi_0 = \pi_1$), MAP $\to$ ML and the ML bit demodulator specified:

$$\begin{aligned} h(t) &= x_1^*(T_p - t) - x_0^*(T_p - t) \\ \gamma &= \frac{1}{2}\big(m_1(T_p) + m_0(T_p)\big) \\ &= \frac{1}{2}\mathrm{Re}\big(E_1 - \sqrt{E_1 E_0}\rho_{10} + \sqrt{E_1 E_0}\rho_{10}^* - E_0\big) \\ &= \frac{1}{2}(E_1 - E_0) \end{aligned}$$

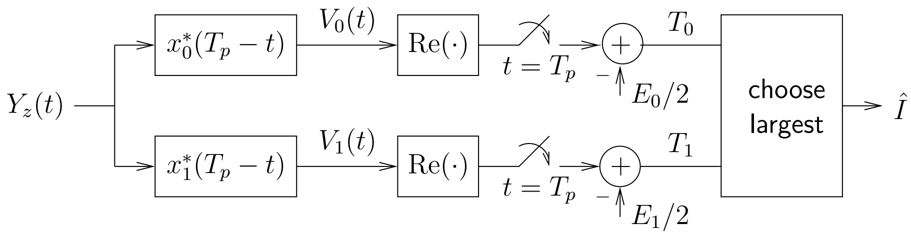
Redraw the block diagram:



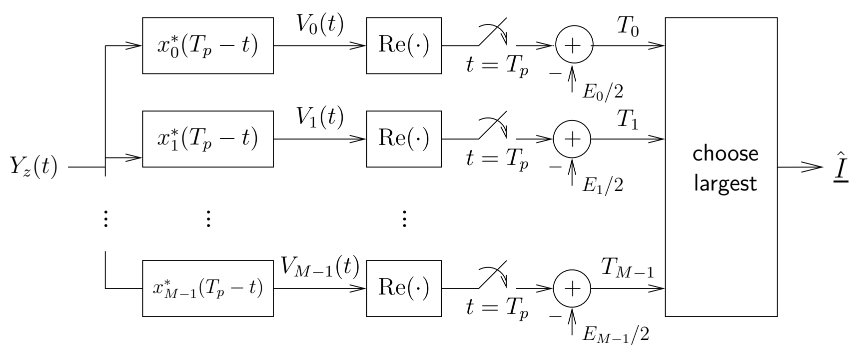
Noting that

$$(v_1 - v_0) \underset{\hat{I}=0}{\overset{\hat{I}=1}{\gtrless}} \frac{E_1 - E_0}{2} \quad \Leftrightarrow \quad (v_1 - \frac{E_0}{2}) \underset{\hat{I}=0}{\overset{\hat{I}=1}{\gtrless}} (v_0 - \frac{E_1}{2}) :$$

can redraw again:



Generalizing to M-ary equal-priors case:



$$V_i(t) \triangleq Y_z(t) * x_i^*(T_p - t) = \int_{-\infty}^{\infty} Y_z(\tau) x_i^*(T_p - t + \tau) d\tau.$$

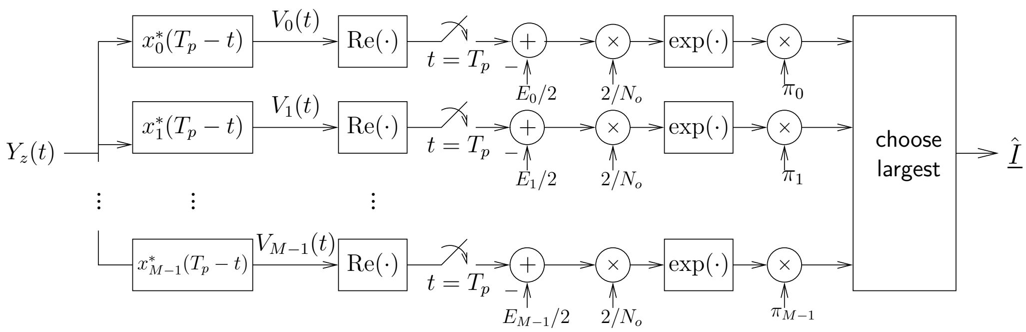
$$T_i \triangleq \mathrm{Re} \int_{-\infty}^{\infty} Y_z(\tau) x_i^*(\tau) d\tau - \frac{E_i}{2}, \quad \text{"ML metric for } I = i\text{"}$$

$$\underline{\hat{I}} = \arg \max_{i \in \{0,\ldots,M-1\}} T_i, \quad \text{"ML word demodulator (MLWD)"}$$

For M-ary non-uniform priors:

$$\underline{\hat{I}} \;=\; \arg\max_{i\in\{0,\dots,M-1\}} \exp\left(\frac{2T_i}{N_0}\right)\pi_i, \qquad \text{“MAPWD”}$$

which yields a similar structure:



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Word Error Probability (WEP) Analysis:

$$\text{WEP} \;\triangleq\; \sum_{j=0}^{M-1} \Pr(\underline{\hat{I}} \neq j \,|\, \underline{I}=j)\pi_j$$

Due to non-linearity of MAPWD, difficult to evaluate in the case of nonuniform priors. So consider $\pi_i = \frac{1}{M}$ and MLWD.

$$\text{WEP} \;=\; \frac{1}{M}\sum_{j=0}^{M-1} \Pr(\underline{\hat{I}} \neq j \,|\, \underline{I}=j)$$

$$\Pr(\underline{\hat{I}} \neq j \,|\, \underline{I}=j) \;=\; \Pr\Big(\max_{i\neq j} T_i > T_j \,|\, \underline{I}=j\Big)$$

What are the statistics of $T_i\big|_{\underline{I}=j}$?

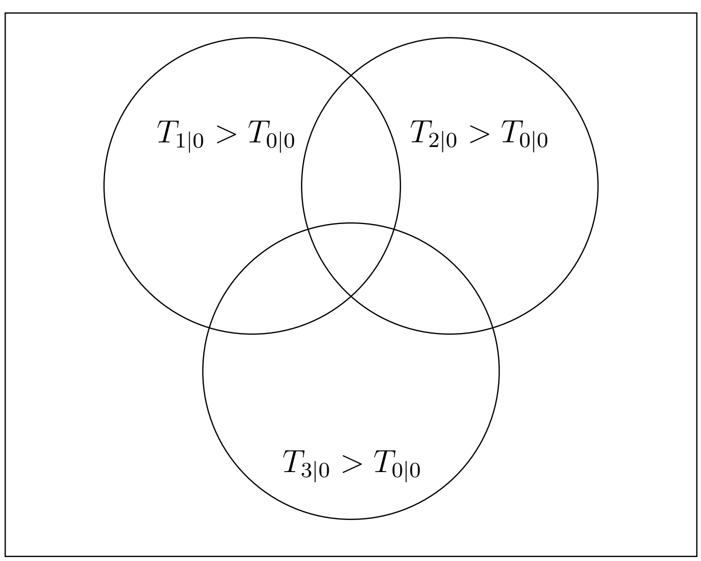
Denote conditional likelihood statistic $T_{i|j} \triangleq T_i\big|_{\underline{I}=j}$.

$$\begin{aligned} T_{i|j} &= \operatorname{Re}\int_{-\infty}^{\infty}(x_j(t)+W_z(t))x_i^*(t)dt - \frac{E_i}{2} \\ &= \sqrt{E_iE_j}\operatorname{Re}\rho_{ji} + N_I^{(i)} - \frac{E_i}{2} \end{aligned}$$

where $N_I^{(i)} \sim \mathcal{N}(0, E_iN_o/2)$. Then using the union bound,

$$\begin{aligned} \Pr(\hat{\underline{I}} \neq j|\underline{I}=j) &= \Pr(\max_{i\neq j} T_{i|j} > T_{j|j}) \\ &= \Pr(\cup_{i=0,i\neq j}^{M-1}\{T_{i|j} > T_{j|j}\}) \\ &\leq \sum_{i=0,i\neq j}^{M-1} \underbrace{\Pr(T_{i|j} > T_{j|j})}_{\text{pairwise error prob.}} \end{aligned}$$

Illustration of union bound (for $j=0$, $M=4$):



The union bound is tight at high SNR.

Analyze pairwise error prob:

$$\begin{aligned}
\Pr(T_{i|j} > T_{j|j}) &= \Pr\left(\sqrt{E_i E_j}\,\mathrm{Re}\,\rho_{ji} - \frac{E_i}{2} + N_{\mathrm{I}}^{(i)} > \frac{E_j}{2} + N_{\mathrm{I}}^{(j)}\right) \\
&= \Pr\Bigg(\underbrace{N_{\mathrm{I}}^{(i)} - N_{\mathrm{I}}^{(j)}}_{N_{ij}} > \underbrace{\frac{E_i + E_j}{2} - \sqrt{E_i E_j}\,\mathrm{Re}\,\rho_{ji}}_{\Delta_E(i,j)/2}\Bigg) \\
N_{ij} &= \mathrm{Re}\int_{-\infty}^{\infty} W_z(t)\big[x_i^*(t) - x_j^*(t)\big]\,dt \\
\sigma_{N_{ij}}^2 &= \frac{N_o}{2}\Delta_E(i,j) \\
\Pr(T_{i|j} > T_{j|j}) &= \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{\Delta_E(i,j)}{4N_o}}\right)
\end{aligned}$$

...same as BEP.

Plug back into union bound:

$$\text{WEP} \;\le\; \frac{1}{M}\sum_{j=0}^{M-1}\sum_{i=0,i\neq j}^{M-1}\frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{\Delta_E(i,j)}{4N_o}}\right)$$

Notes:

- WEP usually dominated by the largest $\Delta_E(i,j)$.
- Frequency of different $\Delta_E(i,j)$: “distance spectrum.”
- Signals with equal $\Delta_E(i,j)$: “geometrically uniform.”
- Signal design: minimize the maximum $\Delta_E(i,j)$.

Optimum Bit Demodulation:

- In some applications, care more about bit error rate.
- Minimizing word error rate $\neq$ minimizing bit error rate!

Goal:

Demodulate $I^{(k)}$ in $\underline{I} = [I^{(1)}, I^{(2)}, \ldots, I^{(K_b)}]$.

MAPBD compares *a posteriori probabilities* (APPs):

$$\Pr\big(I^{(k)} = 1 | y_z(t)\big) \underset{\hat{I}=0}{\overset{\hat{I}=1}{\gtrless}} \Pr\big(I^{(k)} = 0 | y_z(t)\big)$$

but $\Pr\big(y_z(t) | I^{(k)} = m\big)$ is not Gaussian, so we need a different approach.

Define $\underline{I}^{(-)} = [I^{(1)}, \ldots, I^{(k-1)}, I^{(k+1)}, \ldots, I^{(K_b)}]$ and represent $\underline{I}^{(-)} = n \in \{0, 1, \ldots, M/2 - 1\}$.

$$\begin{aligned}\Pr\big(I^{(k)} = m | y_z(t)\big) &= \sum_{n=0}^{M/2-1} \Pr\big(\underbrace{I^{(k)} = m, \underline{I}^{(-)} = n}_{\underline{I} = \{m,n\}} | y_z(t)\big) \\ &= C \sum_{n=0}^{M/2-1} \exp\left(\frac{2T_{\{m,n\}}}{N_o}\right) \pi_{\{m,n\}}\end{aligned}$$

MAPBD:

$$\sum_{n=0}^{M/2-1} \exp\left(\frac{2T_{\{1,n\}}}{N_o}\right) \pi_{\{1,n\}} \underset{\hat{I}=0}{\overset{\hat{I}=1}{\gtrless}} \sum_{n=0}^{M/2-1} \exp\left(\frac{2T_{\{0,n\}}}{N_o}\right) \pi_{\{0,n\}}$$

MLWD:

- Compute $\{T_i\}_{i=0}^{M-1}$.
- Complexity $\mathcal{O}(2^{K_b})$.

MLBD:

- Compute $\{e^{2T_i/N_o}\}_{i=0}^{M-1}$ then average for each bit.
- Complexity $\mathcal{O}(K_b 2^{K_b})$.
- Small improvement in BEP (very small at high SNR).
- Useful in (suboptimum) iterative demodulation schemes!

Example 1: M-ary FSK (with $2f_d$ separation)

$$x_i(t) = \begin{cases} \sqrt{\frac{K_b E_b}{T_p}} \exp\bigl(j2\pi f_d(2i - M + 1)t\bigr) & t \in [0, T_p] \\ 0 & t \notin [0, T_p] \end{cases}$$

where f_d is a design parameter.

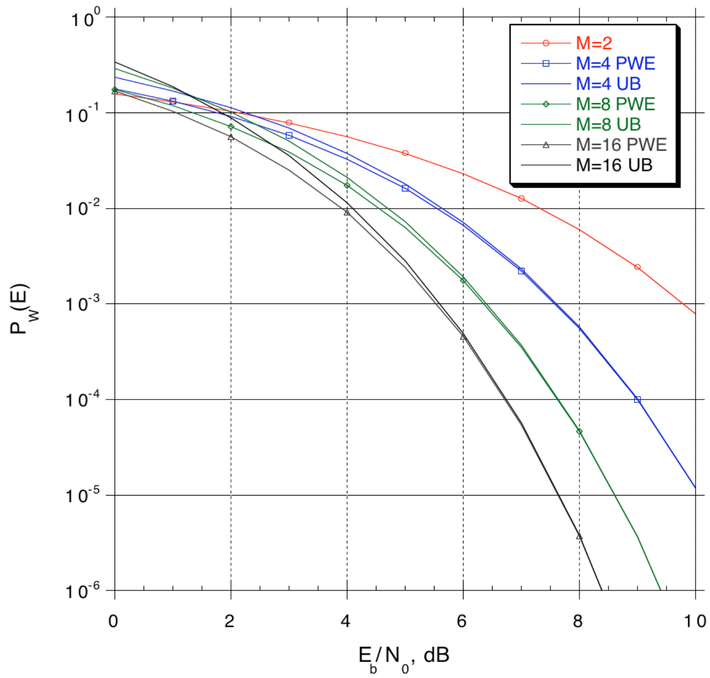
Orthogonal M-FSK:

$f_d = \frac{1}{4T_p} \rightsquigarrow \operatorname{Re} \rho_{ij} = 0$; geometrically uniform.

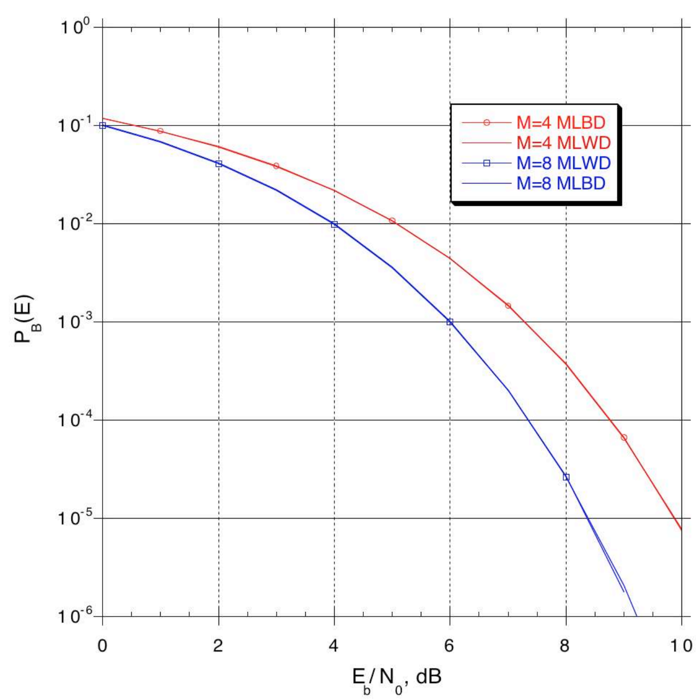
Bandwidth efficiency:

$$B_T = \frac{2^{K_b+1}}{4T_p} \text{ Hz}, \quad W_B = \frac{K_b}{T_p} \frac{\text{bits}}{\text{sec}}, \quad \eta_B = \frac{W_B}{B_T} = \frac{K_b}{2^{K_b-1}}$$

Orthogonal M-FSK: WEP and Union Bound



Orthogonal M-FSK: BEP of MLWD versus MLBD



Orthogonal M-FSK Summary:

As K_b increases:

- reliability increases
- bandwidth efficiency decreases (exponentially)
- complexity increases (exponentially)

Hence, M-FSK useful when high reliability is required and when low spectral efficiency can be tolerated.

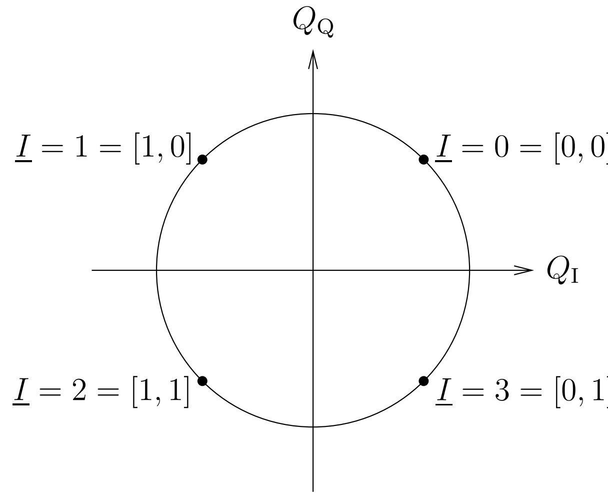
Example 2: M-ary PSK (with uniformly distributed phases)

$$x_i(t) = \begin{cases} \sqrt{\frac{K_b E_b}{T_p}} \exp\left(j\frac{\pi(2i+1)}{M}\right) & t \in [0, T_p] \\ 0 & t \notin [0, T_p] \end{cases}$$

Note common pulse shape: $x_i(t) = d_i u(t)$.

$$\begin{aligned} \underline{\hat{I}} &= \arg\max_i T_i \\ &= \arg\max_i \operatorname{Re}\left[\exp\left(-j\frac{\pi(2i+1)}{M}\right) \underbrace{\int_{-\infty}^{\infty} Y_z(t) u^*(t) dt}_{\text{MF output } Q}\right] \\ &= \arg\min_i \left| \frac{\pi(2i+1)}{M} - \angle Q \right| \end{aligned}$$

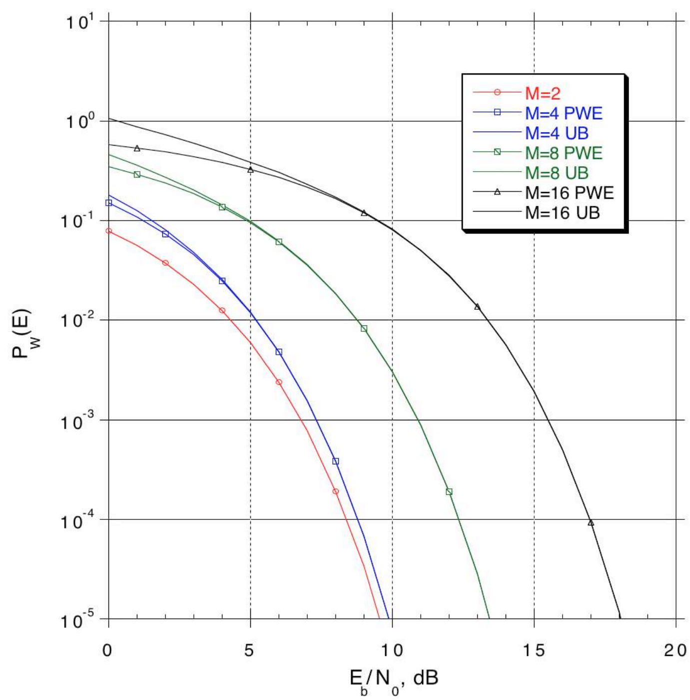
Decision regions for matched-filter output $Q \in \mathbb{C}$:



Gray mapping ensures one bit change for phase neighbors.

Note decoding of $I^{(0)}$ independent of $I^{(1)}$: parallel decoders!

M-PSK: WEP and Union Bound



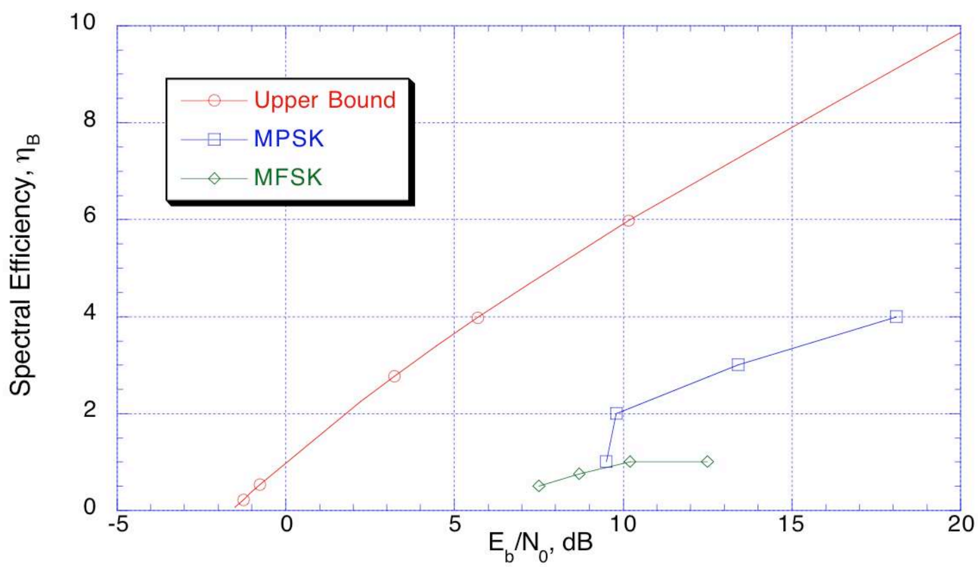
M-PSK Summary:

As K_b increases:

- reliability decreases
- bandwidth efficiency increases ($\eta_B = K_b$)
- complexity doesn't change much

Hence, M-PSK useful when high bandwidth efficiency is required and SNR is reasonably high.

Comparison of spectral efficiencies to Shannon bound:



(considering WEP$= 10^{-5}$ as "reliable")