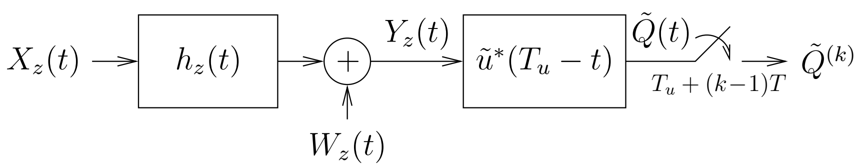
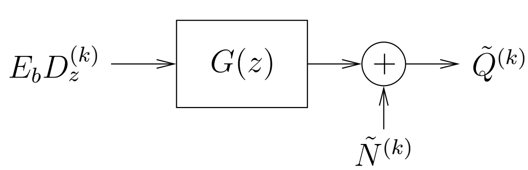
Linear Stream Equalization [Ch. 12]:

- MLWD of stream modulation in time-dispersive (i.e., frequency selective) channels can be accomplished with complexity $\mathcal{O}(K_b N_u 2^{N_u})$, where $N_u = (T_u + T_h)/T$. But when N_u is large, this becomes infeasible.
- Linear equalization presents a sub-optimal, but practical, alternative to MLWD.
- Assume front-end matched-filtering that yields $\{\tilde{Q}^{(k)}\}$:



Discrete-time effective channel:

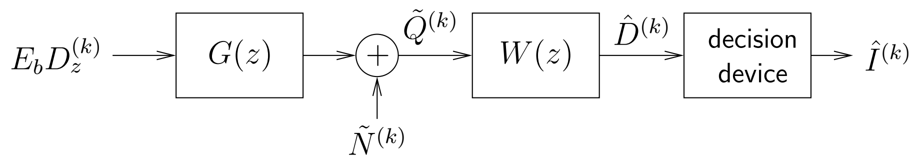
$$\begin{aligned}\tilde{Q}^{(k)} &= \sum_{l=1}^{K_b} D_z^{(l)} \underbrace{V_{\tilde{u}}\big((k-l)T\big)}_{E_b g[k-l]} + \tilde{N}^{(k)} \\ &= E_b \sum_{m=-N_u}^{N_u} g[m] D_z^{(k-m)} + \tilde{N}^{(k)}\end{aligned}$$



Properties:

- $g[0] = 1$ and $g[k] = g^*[-k]$.
- $R_{\tilde{N}}(m) = E_b N_o g[m] \quad \rightsquigarrow \quad S_{\tilde{N}}(e^{j2\pi f}) = E_b N_o G(e^{j2\pi f})$

Linear equalization:



1. Zero-forcing: completely suppress ISI in $\hat{D}^{(k)}$.
2. Linear MMSE: minimize noise+ISI power in $\hat{D}^{(k)}$.

Notes:

- Above equalizers analogous to OCDM linear detectors.
- For now we ignore practical aspects like causality.

Zero-Forcing Equalization:

We formulate in the mixed time/z-domain:

$$\begin{aligned}\hat{D}^{(k)} &= W(z)\tilde{Q}^{(k)} \\ &= W(z)\left[G(z)E_b D_z^{(k)} + \tilde{N}^{(k)}\right]\end{aligned}$$

Thus $W(z) = G^{-1}(z)$ achieves ZF equalization:

$$\hat{D}^{(k)} = E_b D_z^{(k)} + \underbrace{G^{-1}(z)\tilde{N}^{(k)}}_{N_{\text{zf}}^{(k)}}$$

where the effective noise obeys

$$\begin{aligned}S_{N_{\text{zf}}}(e^{j2\pi f}) &= E_b N_o G^{-1}(e^{j2\pi f}) \\ \sigma^2_{N_{\text{zf}}} &= \int_{-0.5}^{0.5} S_{N_{\text{zf}}}(e^{j2\pi f})df \;=\; E_b N_o \langle G^{-1}\rangle_{\text{A}}\end{aligned}$$

For BPSK,

$$\begin{aligned}
\text{PBE} &= \Pr(\hat{D}^{(k)} < 0 | D_z^{(k)} = 1) \ \ \text{since symmetric} \\
&= F_{\hat{D}^{(k)}}(0) \ \ \text{for } \hat{D}^{(k)} \sim \mathcal{N}\Big(E_b, \tfrac{1}{2}E_b N_o \langle G^{-1}\rangle_{\mathsf{A}}\Big) \\
&= \frac{1}{2} + \frac{1}{2}\,\mathrm{erf}\left(\frac{0 - m_{\hat{D}^{(k)}}}{\sqrt{2}\sigma_{\hat{D}^{(k)}}}\right) \\
&= \frac{1}{2} + \frac{1}{2}\,\mathrm{erf}\left(\frac{-E_b}{\sqrt{E_b N_o \langle G^{-1}\rangle_{\mathsf{A}}}}\right) \\
&= \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{E_b}{N_o \langle G^{-1}\rangle_{\mathsf{A}}}}\right)
\end{aligned}$$

Flat G yields $\langle G^{-1}\rangle_{\mathsf{A}} = 1$, but FS G may yield $\langle G^{-1}\rangle_{\mathsf{A}} \gg 1$!

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Linear MMSE Equalization:

Goal: Minimize $\mathrm{E}\{|\hat{D}^{(k)} - E_b D^{(k)}|^2\}$ via choice of $\{w[l]\}$ in $\hat{D}^{(k)} = \sum_{l=-\infty}^{\infty} w[l]\tilde{Q}^{(k-l)}$.

Solve via <u>orthogonality condition</u> of MMSE estimation error:

$$\begin{aligned}
\forall m,\ 0 &= \mathrm{E}\big\{(\hat{D}^{(k)} - E_b D_z^{(k)})\tilde{Q}^{(k-m)*}\big\} \\
&= \sum_l w[l] \underbrace{\mathrm{E}\big\{\tilde{Q}^{(k-l)}\tilde{Q}^{(k-m)*}\big\}}_{R_{\tilde{Q}}[m-l]} - E_b \underbrace{\mathrm{E}\big\{D_z^{(k)}\tilde{Q}^{(k-m)*}\big\}}_{R_{D\tilde{Q}}[m]} \\
\Leftrightarrow \forall f,\ 0 &= W(e^{j2\pi f})S_{\tilde{Q}}(e^{j2\pi f}) - E_b S_{D\tilde{Q}}(e^{j2\pi f})
\end{aligned}$$

which yields

$$W(e^{j2\pi f}) = E_b \frac{S_{D\tilde{Q}}(e^{j2\pi f})}{S_{\tilde{Q}}(e^{j2\pi f})}$$

If $\{D_z^{(k)}\}$ is zero-mean, white, and uncorrelated with $\{\tilde{N}^{(k)}\}$,

$$\begin{aligned} R_{D\tilde{Q}}[m] &= E_b g^*[-m] \;=\; E_b g[m] \\ R_{\tilde{Q}}[m] &= E_b^2 \sum_l g[l]g[m-l] + N_o E_b g[m] \end{aligned}$$

so that

$$\begin{aligned} S_{D\tilde{Q}}(e^{j2\pi f}) &= E_b G(e^{j2\pi f}) \\ S_{\tilde{Q}}(e^{j2\pi f}) &= E_b^2 G^2(e^{j2\pi f}) + N_o E_b G(e^{j2\pi f}) \end{aligned}$$

implying the L-MMSE equalizer

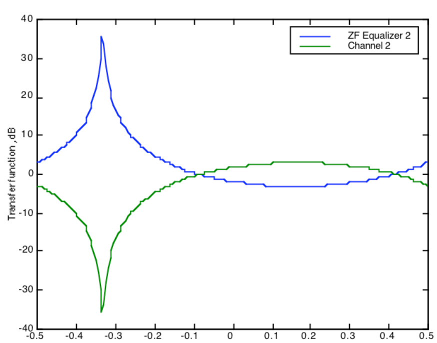
$$W(e^{j2\pi f}) = E_b \frac{S_{D\tilde{Q}}(e^{j2\pi f})}{S_{\tilde{Q}}(e^{j2\pi f})} = \left(G(e^{j2\pi f}) + \frac{N_o}{E_b}\right)^{-1}$$

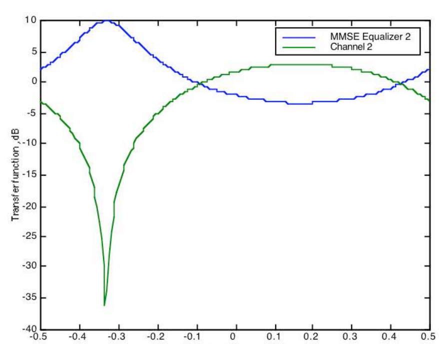
Note: L-MMSE equalizer $\rightarrow$ ZF equalizer as $\frac{N_o}{E_b} \rightarrow 0$.

Unfortunately, PBE analysis for L-MMSE is difficult due to the non-Gaussian nature of the residual ISI.

Example: Channel with deep null

$$h(t) = \sqrt{0.51}\delta(t) + 0.7e^{j2\pi/3}\delta(t-T)$$





Note noise amplification of ZF equalizer! Results in an SNR loss of about 17 dB relative to the flat channel case.