Coded Modulation [Ch. 13]:

- Orthogonal modulation had $\mathcal{O}(K_b)$ complexity MLWD but performance no better than BPSK.
- To improve performance, map *sequence* of info bits onto *sequence* of symbols, then transmit using linear stream modulation. If clever, still have $\mathcal{O}(K_b)$ MLWD.
- Fitz calls it "orthogonal modulation with memory."
- This idea subsumes most coding+modulation schemes.
- We focus on performance, spectral efficiency, and demodulator design rather than on code design.

Basic Idea:

- A sequence of K_b bits $\{I^{(l)}\}_{l=1}^{K_b}$ is mapped to a sequence of N_f constellation labels $\{J^{(l)}\}_{l=1}^{N_f}$.
- Each label $J^{(l)}$ is mapped to symbol $\tilde{D}_z^{(l)} = a(J^{(l)})$.
- The symbol sequence $\{\tilde{D}_z^{(l)}\}_{l=1}^{N_f}$ is linearly modulated using M_s-ary stream modulation.

Fundamental Goals:

- Out of $M_s^{N_f}$ possible symbol sequences, choose 2^{K_b} sequences with good Euclidean distance properties.
- Ensure that the bit-sequence to symbol-sequence mapping allows $\mathcal{O}(K_b)$ MLWD. (Idea: use FSM.)

Outline:

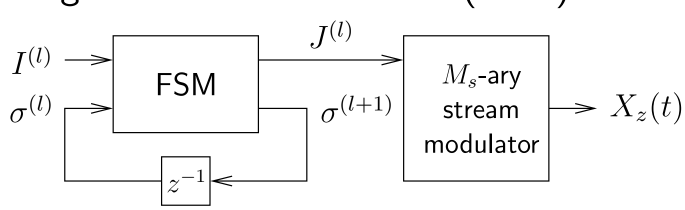
1. Rate-1 mappings (i.e., $N_f \approx K_b$).
2. Arbitrary rate mappings: convolutional and trellis codes.
3. Duality between codes and frequency-selective channels.
4. $\mathcal{O}(K_b)$ demodulation [Ch. 14].

Assumptions:

- Bits $\{I^{(l)}\}_{l=1}^{K_b}$ are independent and equally likely.
- Symbol mapping ensures $\mathrm{E}\left[|\tilde{D}_z^{(l)}|^2\right] = R$ (i.e., rate).

Coded Modulation for $R = 1$:

- K_b bits $\{I^{(l)}\}$ mapped onto K_b constellation labels $\{J^{(l)}\}$ using a *finite state machine* (FSM).

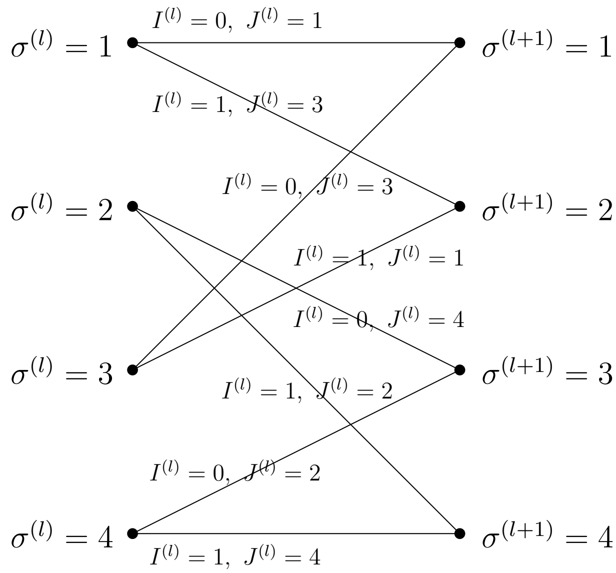


- FSM characterized by N_s modulation states $\sigma^{(l)} \in \Omega_\sigma$:

$$\begin{aligned} \sigma^{(l+1)} &= g_1\big(\sigma^{(l)}, I^{(l)}\big) \\ J^{(l)} &= g_2\big(\sigma^{(l)}, I^{(l)}\big) \end{aligned}$$

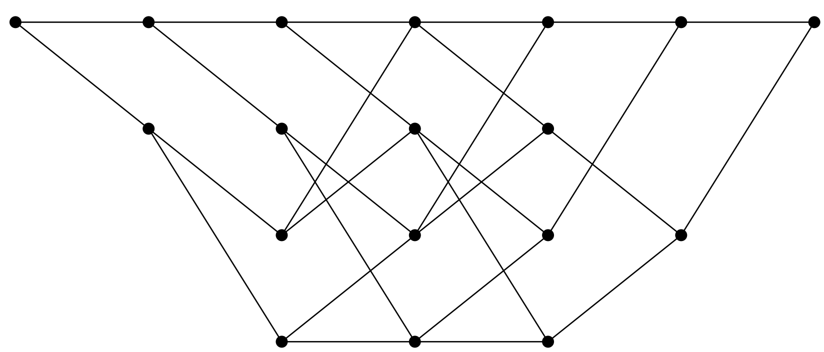
  Larger N_s means more freedom in sequence design but higher demod complexity.

FSM well described by a *trellis diagram*:



This Ho-Cavers-Varaldi trellis code has $N_s = 4$ and $M_s = 4$.

Expanded trellis diagram for $K_b = 4$ bits:



Here, $\nu_c = 2$ extra bits are used to return to initial state ("termination"). Thus, for $R = 1$, have frame length $N_f = K_b + \nu_c$. Note $R_{\text{eff}} = \frac{K_b}{K_b + \nu_c} \approx 1 = R$ for large K_b.

Also note: $\#$ of paths through trellis $= 2^{K_b} = 16$.

<u>MLWD</u>:

Orthogonal modulation leads to a decoupled ML metric:

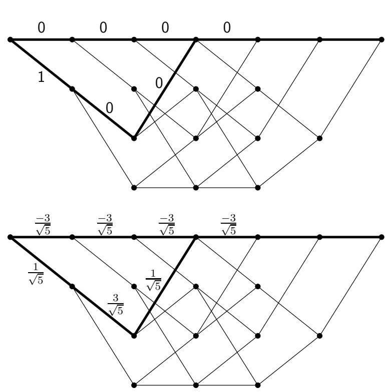
$$\begin{aligned}\underline{\hat{I}} &= \arg \max_{i \in \{0,\dots,2^{K_b}-1\}} T_i \\ &= \arg\max_i \sqrt{E_b} \sum_{k=1}^{N_f} \mathrm{Re}\big[\tilde{d}_i^{(k)*} Q^{(k)}\big] - \frac{E_b}{2} \sum_{k=1}^{N_f} |\tilde{d}_i^{(k)}|^2 \\ &= \arg\min_i \sum_{k=1}^{N_f} \Big| Q^{(k)} - \sqrt{E_b}\, \tilde{d}_i^{(k)} \Big|^2\end{aligned}$$

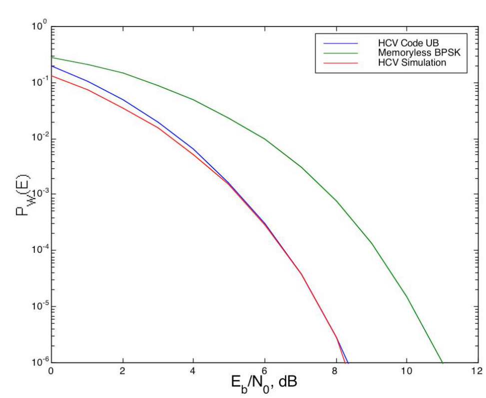
Hence, MLWD $\Leftrightarrow$ Minimizing Euclidean sequence norm.

$$\Delta_E(i,j) \;=\; \int |x_i(t) - x_j(t)|^2 dt \;=\; E_b \sum_{k=1}^{N_f} \Big| \tilde{d}_i^{(k)} - \tilde{d}_j^{(k)} \Big|^2$$

Example: HCV code with 4-PAM modulation and $K_b = 4$.

- $J^{(l)} \in [1,2,3,4] \quad \leftrightarrow \quad \tilde{D}_z^{(l)} \in \left[\frac{-3}{\sqrt{5}}, \frac{-1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{3}{\sqrt{5}}\right]$
- $\frac{1}{2} 2^{K_b}(2^{K_b} - 1) = 160$ distances in error spectrum.
- $\Delta_E\big([0000],[1000]\big) = 13.6 E_b \gg 4E_b$ (recall BPSK).





Spectral Characteristics:

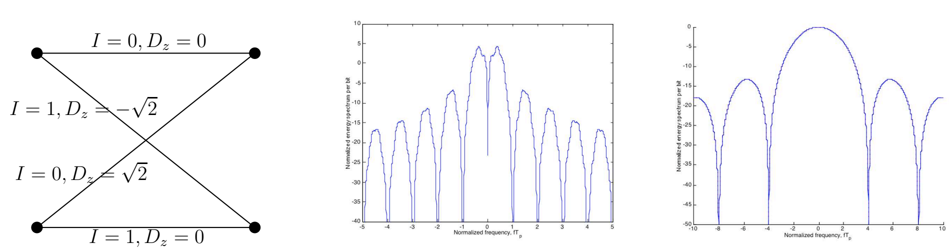
- Though the info bits $\{I^{(k)}\}$ are iid, the coded symbols $\{\tilde{D}_z^{(l)}\}$ will be correlated.
- Correlated symbols lead to a "shaping" of the power spectrum.
- In some cases, the spectrum becomes more compact, which is reason enough to use modulation with memory.
- In the sequel, we develop tools to analyze the spectrum.

Energy spectrum (averaged per bit):

$$
\begin{aligned}
D_{X_z}(f) &= \frac{E_b}{K_b} \mathrm{E}\left[|X_z(f)|^2\right] \\
&= \frac{E_b}{K_b 2^{K_b}} \sum_{i=0}^{2^{K_b}-1} \left| \sum_{k=1}^{N_f} \tilde{d}_i^{(k)} U(f) e^{j2\pi f T(k-1)} \right|^2
\end{aligned}
$$

Possible to compute above equation for small K_b.

Example: Alternate Mark Inversion with $K_b = 4$:



In the case of large K_b, need a different approach...

$$\begin{aligned}
D_{X_z}(f) &= \frac{E_b}{K_b}\,\mathrm{E}\left|\sum_{l=1}^{N_f} \tilde{D}_z^{(l)} U(f) e^{-j2\pi f T(l-1)}\right|^2 \\
&= \frac{E_b}{K_b}\sum_{l=1}^{N_f}\sum_{k=1}^{N_f} \underbrace{\mathrm{E}\left[\tilde{D}_z^{(l)}\tilde{D}_z^{(k)*}\right]}_{R_{\tilde{D}}[l-k]} |U(f)|^2 e^{-j2\pi f T(l-k)} \\
&= E_b|U(f)|^2 \frac{1}{K_b}\sum_{m=-N_f+1}^{N_f-1} (N_f-|m|) R_{\tilde{D}}[m] e^{-j2\pi f T m} \\
&= E_b|U(f)|^2 \sum_{m=-\infty}^{\infty} R_{\tilde{D}}[m] e^{-j2\pi f T m} \quad \text{as } K_b \to \infty \\
&= E_b|U(f)|^2 S_{\tilde{D}}(e^{j2\pi f T})
\end{aligned}$$

To find $R_{\tilde{D}}[m]$, note

$$R_{\tilde{D}}[m] = \sum_{d_i}\sum_{d_j} d_i d_j^* P_{\tilde{D}_z^{(l)},\tilde{D}_z^{(l-m)}}(d_i, d_j)$$

- The trellis edge $S^{(l)}$ connecting state $\sigma^{(l)}$ to $\sigma^{(l+1)}$ completly determines the symbol $\tilde{D}_z^{(l)}$.
- Can represent $S^{(l)}$ by an integer in $\{1,\ldots,2N_s\}$.

Thus we note that

$$P_{S^{(l)},S^{(l-m)}}(s_i,s_j) \;\rightarrow\; P_{\tilde{D}_z^{(l)},\tilde{D}_z^{(l-m)}}(d_i,d_j)$$

To characterize $P_{S^{(l)},S^{(l-m)}}(\cdot,\cdot)$, we use the fact that

$$P_{S^{(l)},S^{(l-m)}}(s_i,s_j) = P_{S^{(l)}|S^{(l-m)}}(s_i|s_j) P_{S^{(l-m)}}(s_j)$$

Assume uniform $P_{S^{(l-m)}}(\cdot)$. To find $P_{S^{(l)}|S^{(l-m)}}(\cdot|\cdot)$, note

$$P_{S^{(l)}}(s_i) = \sum_{s_j=1}^{2N_s} P_{S^{(l)},S^{(l-1)}}(s_i, s_j) = \sum_{s_j=1}^{2N_s} \underbrace{P_{S^{(l)}|S^{(l-1)}}(s_i|s_j)}_{\triangleq [\boldsymbol{S}_T]_{j,i}} P_{S^{(l-1)}}(s_j)$$

where $[\boldsymbol{S}_T]_{j,i}$ are easily determined from $g_1(I^{(l)}, \sigma^{(l)})$.

Defining pmf vector $\underline{P}_{S^{(l)}} \triangleq [P_{S^{(l)}}(1), \ldots, P_{S^{(l)}}(2N_s)]$,

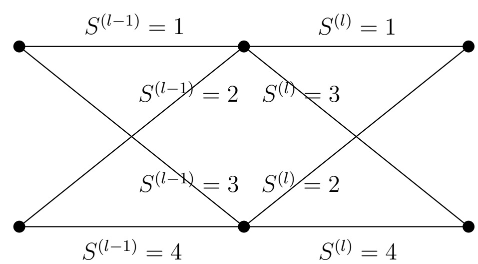
$$\begin{aligned} \underline{P}_{S^{(l)}} &= \underline{P}_{S^{(l-1)}} \boldsymbol{S}_T, \qquad \underline{P}_{S^{(l-1)}} = \underline{P}_{S^{(l-2)}} \boldsymbol{S}_T \\ \Rightarrow \underline{P}_{S^{(l)}} &= \underline{P}_{S^{(l-m)}} \boldsymbol{S}_T^m \end{aligned}$$

From the definition of $[\boldsymbol{S}_T]_{j,i}$ above, we can now see that

$$P_{S^{(l)}|S^{(l-m)}}(s_i|s_j) = [\boldsymbol{S}_T^m]_{j,i}.$$

<u>AMI Example</u>: Assume $\underline{P}_{S^{(l-m)}} = [\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}]$.

$$\boldsymbol{S}_t = \begin{pmatrix} 0.5 & 0 & 0.5 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0 & 0.5 \end{pmatrix}$$



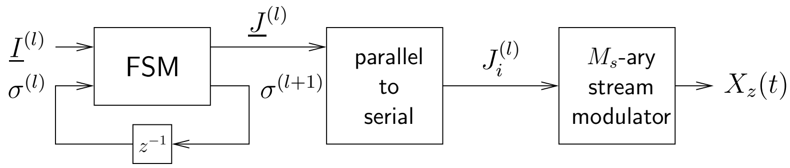
Note that $\boldsymbol{S}_t^m\big|_{m>1}$ has equal entries.

$\rightsquigarrow$ Edges $S^{(l)}$ & $S^{(l-m)}$ independent for $m > 1$.
$\rightsquigarrow$ Symbols $\tilde{D}_z^{(l)}$ & $\tilde{D}_z^{(l-m)}$ uncorrelated for $m > 1$.

Can show that

$$\begin{aligned} \big(R_{\tilde{D}}[-1], R_{\tilde{D}}[0], R_{\tilde{D}}[1]\big) &= \big(\tfrac{1}{2}, 1, -\tfrac{1}{2}\big) \\ \Rightarrow S_{\tilde{D}}(e^{j2\pi fT}) &= 1 - \cos(2\pi fT) \end{aligned}$$

Coded Modulation for General R:



- K_b bits parsed into N_b blocks of K_m bits ($K_b = N_b K_m$)
- The FSM accepts $\underline{I}^{(l)}$, a block of K_m bits, and produces $\underline{J}^{(l)}$, a block of N_m constellation labels.

$$\begin{aligned} \sigma^{(l+1)} &= g_1(\sigma^{(l)}, \underline{I}^{(l)}) \\ \underline{J}^{(l)} &= g_2(\sigma^{(l)}, \underline{I}^{(l)}) \end{aligned}$$

- Label sequence drives M_s-ary linear stream modulation:

$$a\big(J_i^{(l)}\big) = \tilde{D}_z^{((l-1)N_m+i)}$$

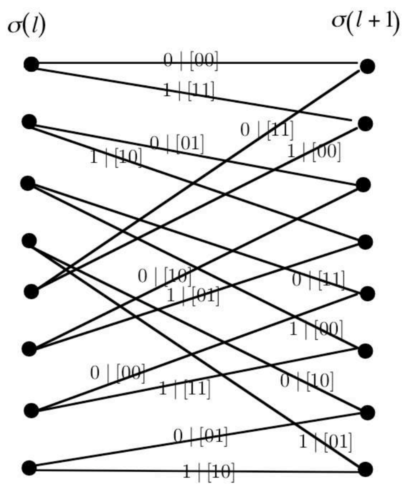
- Total # of symbols in frame is $N_f = N_b N_m + \nu_c$.
- Effective rate (bits/channel-use) is

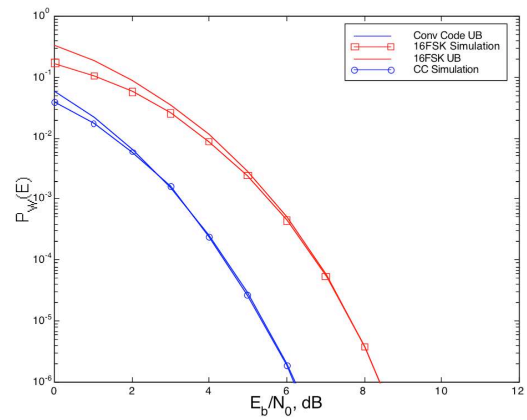
$$R_{\text{eff}} = \frac{K_b}{N_b N_m + \nu_c} = \frac{K_b K_m}{K_b N_m + \nu_c} \approx \frac{K_m}{N_m} \triangleq R.$$

- Might choose $R > 1$ or $R < 1$ depending on desired performance/spectral-efficiency tradeoff.
- Symbols always normalized so that $\mathrm{E}\big|\tilde{D}_z^{(l)}\big|^2 = R$.

$R < 1$ Example: Convolutional Code

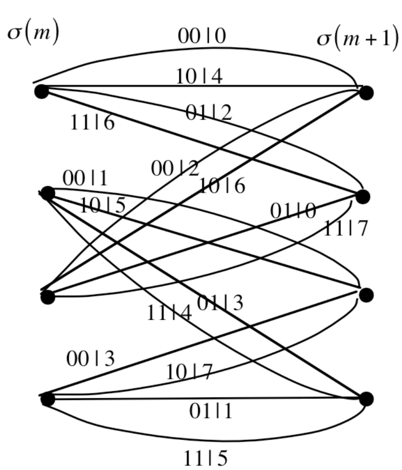
- BPSK ($M_s = 2$), $R = \frac{1}{2}$ ($K_m = 1, N_m = 2$), $N_s = 8$
- ≈2 dB better than 16-FSK at similar spectral efficiency.

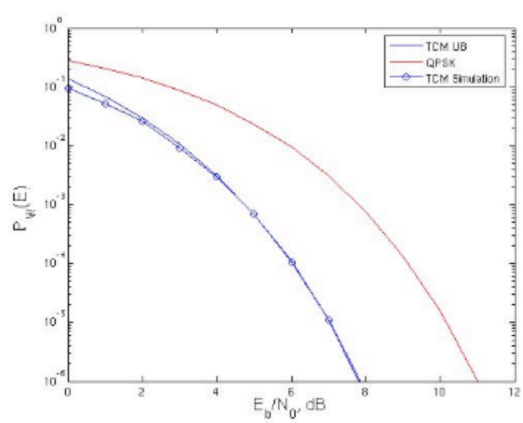




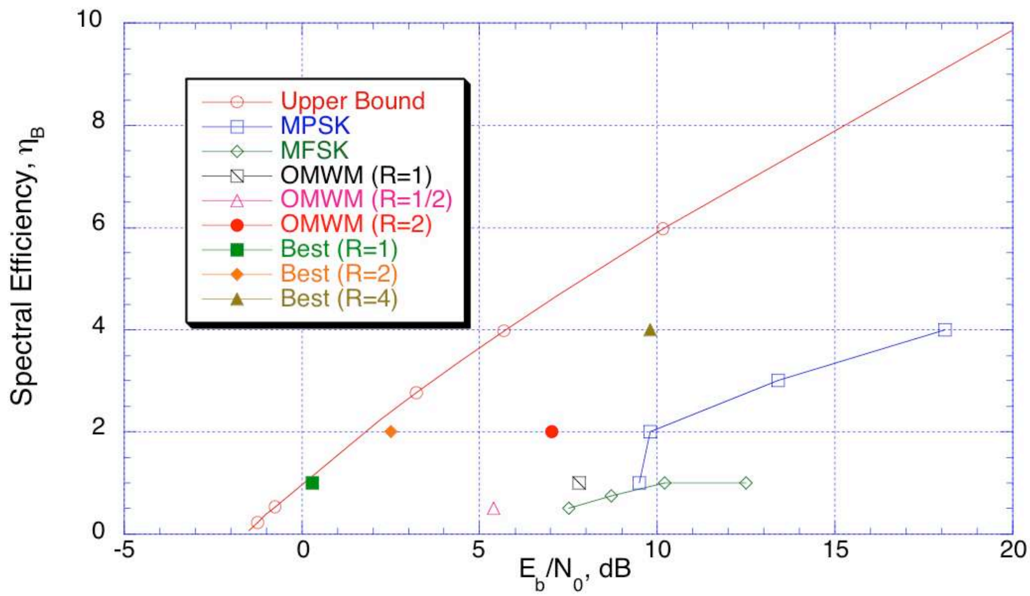
$R > 1$ Example: Trellis Code

- 8-PSK ($M_s = 8$), $R = 2$ ($K_m = 2, N_m = 1$), $N_s = 4$.
- ≈3 dB better than QPSK at same spectral efficiency.



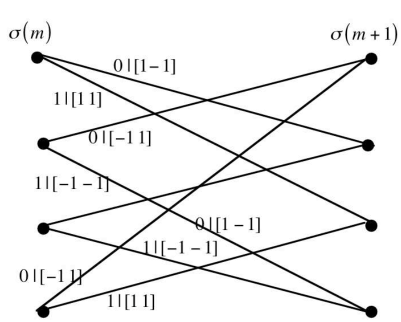


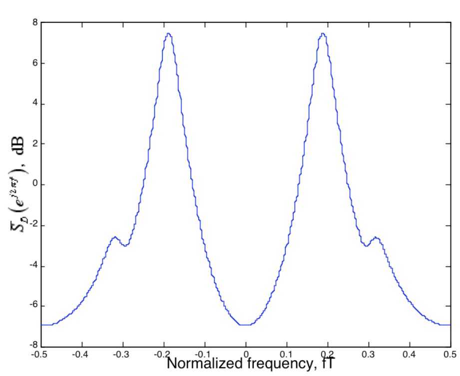
Spectral efficiency of coded modulation schemes:



Spectral Shaping Example: the Miller Code

- $N_s = 4$, BPSK ($M_s = 2$), $R = \frac{1}{2}$ ($K_m = 1, N_m = 2$)
- Run Length Limited: ≤ 4 same symbols in a row.
- Used in magnetic recording, since low frequencies interfere with servo mechanism of read/write head.





Analyzing Spectral Characteristics when $N_m > 1$

$$D_{X_z}(f) = \frac{E_b}{K_b}\sum_{l=1}^{N_f}\sum_{k=1}^{N_f} \mathrm{E}\left[\tilde{D}_z^{(l)}\tilde{D}_z^{(k)*}\right] |U(f)|^2 e^{-j2\pi fT(l-k)}$$

$$\mathrm{E}\left[\tilde{D}_z^{(l)}\tilde{D}_z^{(k)*}\right] = R_{\tilde{D}}\big[l-k, \langle l-1\rangle_{N_m}+1\big] \quad \text{"cyclostationary"}$$

As $K_b \to \infty$, the same techniques used before yield

$$D_{X_z}(f) = E_b|U(f)|^2 \sum_{m=-\infty}^{\infty} \underbrace{\frac{R}{N_m}\sum_{l=1}^{N_m} R_{\tilde{D}}[m,l]\, e^{-j2\pi fTm}}_{\triangleq\, \overline{R}_{\tilde{D}}[m]}$$

$$= E_b|U(f)|^2 \overline{S}_{\tilde{D}}(e^{j2\pi fT})$$

To find $R_{\tilde{D}}[m,l]$, we start with the definition

$$R_{\tilde{D}}[m,l] = \sum_{d_i}\sum_{d_j} d_i d_j^* P_{\tilde{D}_z^{((k-1)N_m+l)},\tilde{D}_z^{((k-1)N_m+l-m)}}(d_i, d_j)$$

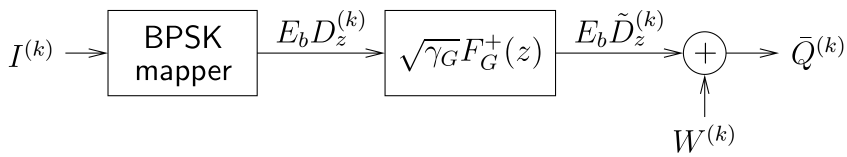
Noting that the edge determines the symbol-block:

$$S^{(k)} \;\to\; \{\tilde{D}_z^{((k-1)N_m+l)}\}_{l=1}^{N_m}$$

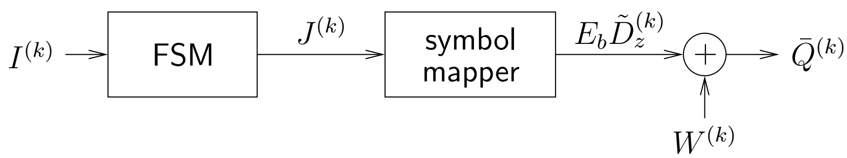
1. Use, as before, $[\boldsymbol{S}_T^m]_{j,i} = P_{S^{(l)}|S^{(l-m)}}(i|j)$ and the uniform $P_{S^{(l-m)}}(\cdot)$ assumption to find $P_{S^{(l)},S^{(l-m)}}(\cdot,\cdot)$.
2. Use $P_{S^{(l)},S^{(l-m)}}(\cdot,\cdot)$ and the trellis description to find $R_{\tilde{D}}[m,l]$. This entails averaging over the multiple symbols per edge.

Channel/Code Duality:

Recall the WMF model for uncoded stream modulation in a frequency selective channel. A $\sqrt{\gamma_G}$-scaled version looks like:



Now consider rate-1 coded stream modulation in a frequency flat channel:



Note the similarities!

Comments:

- Called "Forney equivalence" in Fitz's notes.
- Important implication: same MLWD for both cases.
- With a FS channel, the effective symbols $\{\tilde{D}_z^{(k)}\}$ obey

$$\tilde{D}_z^{(k)} = \sqrt{\gamma_G} \sum_{m=0}^{N_u} f_G[m] D_z^{(k-m)}$$

  hence channel acts as a code with $R = 1$ & 2^{N_u} states.
- In general, channels form "bad" codes: they decrease, rather than increase, the minimum distance!
- N_s-state coded modulation over a FS channel can be interpreted as $N_s 2^{N_u}$-state coding over a flat chan.