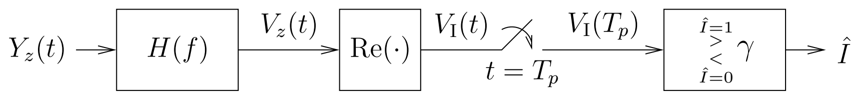
Communication of a Single Bit: [Ch. 7]

Problem Setup:

- $I \in \{0,1\}$
- $\pi_0 \triangleq P(I=0)$ and $\pi_1 \triangleq P(I=1)$ ("priors")
- $I = i \quad \Rightarrow \quad X_z(t) = x_i(t)$ with time support on $[0, T_p]$
- $Y_z(t) = X_z(t) + W_z(t)$ for CWGN with $S_{W_z}(f) = N_o$

To minimize bit error probability (BEP)...

- Use the following structure [App. A, ECE-806]:



- Choose γ, $H(f)$, $\{x_0(t), x_1(t)\}$ in that order.

Hypothesis Testing:

- *Maximum a posteriori* (MAP) detector gives min BEP:

$$\hat{I} = \arg\max_i \Pr\big(I = i \,\big|\, V_{\mathrm{I}}(T_p) = v_{\mathrm{I}}\big)$$

Using Bayes rule

$$\begin{aligned} \Pr\big(I = i \,\big|\, V_{\mathrm{I}}(T_p) = v_{\mathrm{I}}\big) &= \frac{f_{V_{\mathrm{I}}(T_p)|I}(v_I|i)\,\pi_i}{f_{V_{\mathrm{I}}(T_p)}(v_{\mathrm{I}})} \\ \Rightarrow \hat{I} &= \arg\max_i f_{V_{\mathrm{I}}(T_p)|I}(v_I|i)\,\pi_i \end{aligned}$$

So what is $f_{V_{\mathrm{I}}(T_p)|I}(v_I|i)$?

- Note that

$$V_{\rm I}(T_p)\Big|_{I=i} = m_i(T_p) + N_{\rm I}(T_p)$$

where

$$m_i(t) = \operatorname{Re}\int_{-\infty}^{\infty} x_i(\tau)h(t-\tau)d\tau$$

$$N_{\rm I}(T_p) \sim \mathcal{N}(0,\sigma_{N_{\rm I}}^2) \text{ for some } \sigma_{N_{\rm I}}^2$$

thus

$$f_{V_{\rm I}(T_p)|I}(v_{\rm I}, i) = \frac{1}{\sqrt{2\pi\sigma_{N_{\rm I}}^2}}\exp\left[-\frac{\big(v_{\rm I} - m_i(T_p)\big)^2}{2\sigma_{N_{\rm I}}^2}\right]$$

- In the case that $\pi_0 = \pi_1$ ("equal priors"),

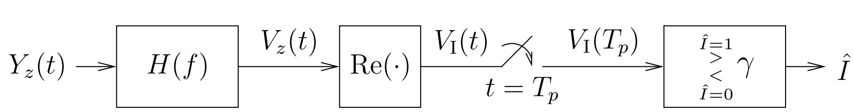
$$\hat{I} = \arg\max_i f_{V_{\rm I}(T_p)|I}(v_{\rm I}|i)$$
$$= \arg\min_i \big(v_{\rm I} - m_i(T_p)\big)^2$$

The optimal detector minimizes Euclidean distance!

This gives the following threshold test:

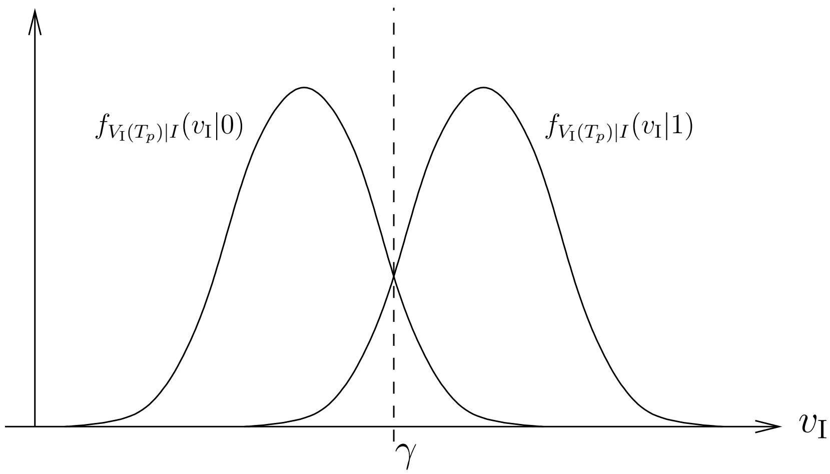
$$V_{\rm I}(T_p) \underset{\hat{I}=0}{\overset{\hat{I}=1}{\gtrless}} \gamma \quad \text{where} \quad \gamma = \frac{m_1(T_p) + m_0(T_p)}{2}$$

which helps to justify the block diagram:



Analysis of bit error probability:

$$\begin{aligned}\text{BEP} &= \Pr(\hat{I} \neq I) \\ &= \Pr(\hat{I} = 1 | I = 0)\pi_0 + \Pr(\hat{I} = 0 | I = 1)\pi_1 \\ &= \Pr\big(V_{\rm I}(T_p) > \gamma | I = 0\big)\pi_0 + \Pr\big(V_{\rm I}(T_p) < \gamma | I = 1\big)\pi_1\end{aligned}$$



- How do we compute Gaussian tail probabilities?

  Say $X \sim \mathcal{N}(\mu, \sigma^2)$. Consider Gaussian CDF:

$$\begin{aligned}F_X(x) &= \Pr(X < x) \\ &= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right) dt \\ &= \frac{1}{2} + \frac{1}{2}\,\mathrm{erf}\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)\end{aligned}$$

  where $\mathrm{erf}(\cdot)$ is a well-tabulated function:

$$\mathrm{erf}(x) \triangleq \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \triangleq 1 - \mathrm{erfc}(x)$$

- For the case that $\pi_0 = \pi_1$, we can use the properties $\gamma = \frac{m_1(T_p)+m_0(T_p)}{2}$ and $V_{\rm I}(T_p)\big|_{I=i} \sim \mathcal{N}(m_i(T_p), \sigma^2_{N_{\rm I}})$:

$$\begin{aligned} P\big(\hat{I}=1|I=0\big) &= \frac{1}{2}\,\mathrm{erfc}\left(\frac{m_1(T_p)-m_0(T_p)}{2\sqrt{2}\,\sigma_{N_{\rm I}}}\right) \\ &= P\big(\hat{I}=0|I=1\big) \\ &= \text{BEP} \end{aligned}$$

Then, defining the <u>effective SNR</u> η:

$$\eta \;\triangleq\; \left(\frac{m_1(T_p)-m_0(T_p)}{2\sqrt{2}\,\sigma_{N_{\rm I}}}\right)^2$$

we get the simple expression

$$\text{BEP} \;=\; \frac{1}{2}\,\mathrm{erfc}(\sqrt{\eta}) \;\approx\; \frac{1}{2}e^{-\eta} \text{ at high } \eta$$

- Note: minimizing BEP $\Leftrightarrow$ maximizing η. From

$$\begin{aligned} \sigma^2_{N_{\rm I}} &= \frac{N_o}{2}\int_{-\infty}^{\infty}|H(f)|^2 df \\ m_1(T_p)-m_0(T_p) &= \mathrm{Re}\left(\mathcal{F}^{-1}\Big\{\big[X_1(f)-X_0(f)\big]H(f)\Big\}_{t=T_p}\right) \\ &= \mathrm{Re}\int_{-\infty}^{\infty}\underbrace{\big[X_1(f)-X_0(f)\big]}_{\triangleq B_{10}(f)}H(f)e^{j2\pi f T_p}df \end{aligned}$$

we find

$$\eta \;=\; \frac{\left(\mathrm{Re}\int_{-\infty}^{\infty}B_{10}(f)H(f)e^{j2\pi fT_p}df\right)^2}{4N_0\int_{-\infty}^{\infty}|H(f)|^2 df}$$

So how can $H(f)$ be chosen to maximize η?

Use Schwarz inequality:

$$\frac{|\int_{-\infty}^{\infty} H(f)Y^*(f)df|^2}{\int_{-\infty}^{\infty} |H(f)|^2 df \cdot \int_{-\infty}^{\infty} |Y(f)|^2 df} \leq 1 \qquad \text{with equality iff } H(f) = CY(f)$$

Setting $Y(f) = B_{10}^*(f)e^{-j2\pi f T_p}$, we can see

$$\begin{aligned} H_{\text{max}}(f) &= CB_{10}^*(f)e^{-j2\pi f T_p} \\ \eta_{\text{max}} &= \frac{1}{4N_o}\int_{-\infty}^{\infty} |B_{10}(f)|^2 df \end{aligned}$$

After choosing $C = 1$ (without loss of optimality), we find

$$\begin{aligned} h_{\text{max}}(t) &= \mathcal{F}^{-1}\left\{B_{10}^*(f)e^{-j2\pi f T_p}\right\} = b_{10}^*(T_p - t) \\ &= x_1^*(T_p - t) - x_0^*(T_p - t) \quad (\text{"effective signal"}) \end{aligned}$$

known as a "matched filter" receiver.

- Finally we design $\{x_o(t), x_1(t)\}$ to maximize η_{max}:

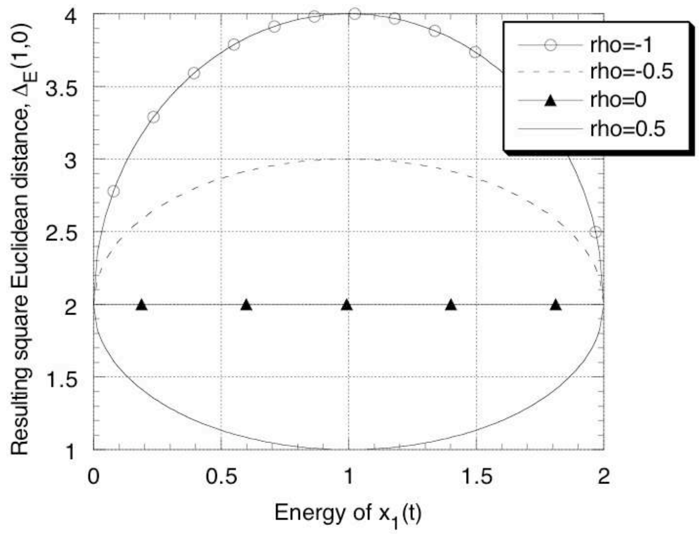
$$\begin{aligned} \eta_{\text{max}} &= \frac{1}{4N_o}\int_{-\infty}^{\infty} |B_{10}(f)|^2 df = \frac{1}{4N_o}\int_{-\infty}^{\infty} |b_{10}(t)|^2 dt \\ &= \frac{1}{4N_o}\underbrace{\int_{-\infty}^{\infty} |x_1(t) - x_0(t)|^2 dt}_{\Delta_E(1,0)} \end{aligned}$$

The Euclidean distance $\Delta_E(1,0)$ obeys

$$\Delta_E(1,0) = \underbrace{\int_{-\infty}^{\infty} |x_1(t)|^2 dt}_{E_1} + \underbrace{\int_{-\infty}^{\infty} |x_0(t)|^2 dt}_{E_0} - 2\,\text{Re}\underbrace{\int_{-\infty}^{\infty} x_1(t)x_0^*(t)dt}_{\sqrt{E_1 E_0}\,\rho_{10}}$$

where $|\rho_{10}| \leq 1$ is the <u>signal correlation coefficient</u>.

The antipodal signaling choices $\rho_{10} = -1$ & $E_0 = E_1 = E_b$ minimize BEP $= \frac{1}{2}\operatorname{erfc}\left(\sqrt{\frac{\Delta_E(1,0)}{4N_0}}\right) = \frac{1}{2}\operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$.



- Example 1: Binary Frequency Shift Keying (BFSK):

$$x_o(t) = \begin{cases} \sqrt{\frac{E_b}{T_p}}\exp(j2\pi f_d t) & t \in [0, T_p] \\ 0 & t \notin [0, T_p] \end{cases}$$

$$x_1(t) = \begin{cases} \sqrt{\frac{E_b}{T_p}}\exp(-j2\pi f_d t) & t \in [0, T_p] \\ 0 & t \notin [0, T_p] \end{cases}$$

where f_d is a design parameter.

Note that $E_0 = E_1 = E_b$ and that

$$\rho_{10} = \frac{1}{T_p}\int_0^{T_p} \exp(-j4\pi f_d t)dt$$

Recall that $\Delta_E(1,0) = E_0 + E_1 - 2\sqrt{E_0 E_1}\operatorname{Re}\rho_{10}$.

Design strategies:

1. Choose f_d to minimize $\operatorname{Re}\rho_{10}$ (i.e., maximize $\Delta_E(1,0)$):

$$f_d \approx \frac{3}{8T_p}, \qquad \operatorname{Re}\rho_{10} \approx -0.21, \qquad \rightsquigarrow \text{2.2dB SNR loss.}$$

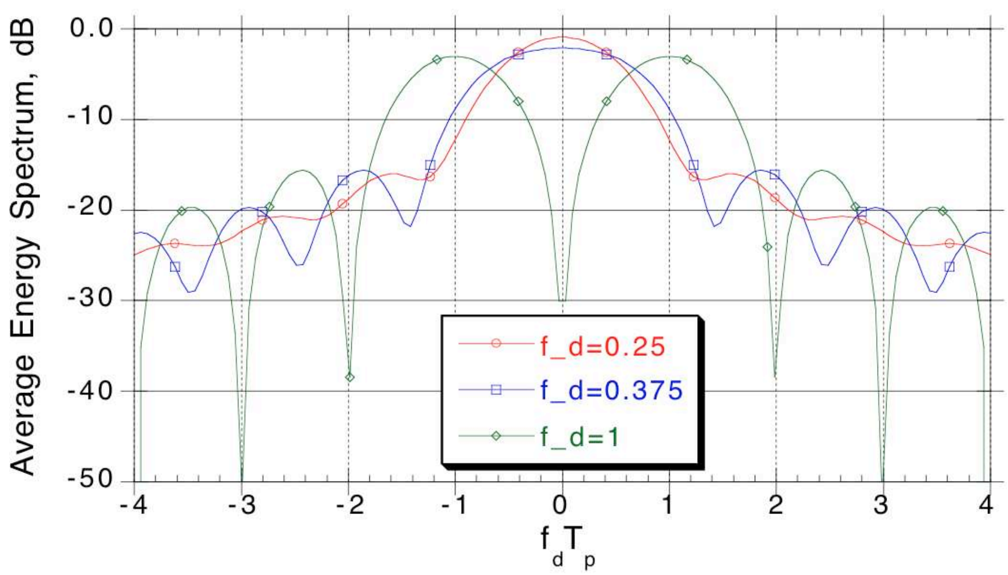
2. Choose f_d to yield orthogonal signals (i.e., $\rho_{10} = 0$):

$$f_d = \frac{1}{4T_p} \text{ ("called MSK")}, \qquad \rightsquigarrow \text{3dB SNR loss.}$$

Spectral efficiency: $W_b = \frac{1}{T_p}$ bits/sec, $B_T \approx \frac{1.5}{T_p}$ Hz.

$$\eta_B \triangleq \frac{W_b}{B_T} \approx 0.67 \text{ bits/sec/Hz}$$

BFSK energy spectrum for various f_d:



- Example 2: Binary Phase Shift Keying (BPSK):

$$x_o(t) = \begin{cases} \sqrt{\frac{E_b}{T_p}} & t \in [0, T_p] \\ 0 & t \notin [0, T_p] \end{cases}$$

$$x_1(t) = \begin{cases} \sqrt{\frac{E_b}{T_p}} \exp(j\theta) & t \in [0, T_p] \\ 0 & t \notin [0, T_p] \end{cases}$$

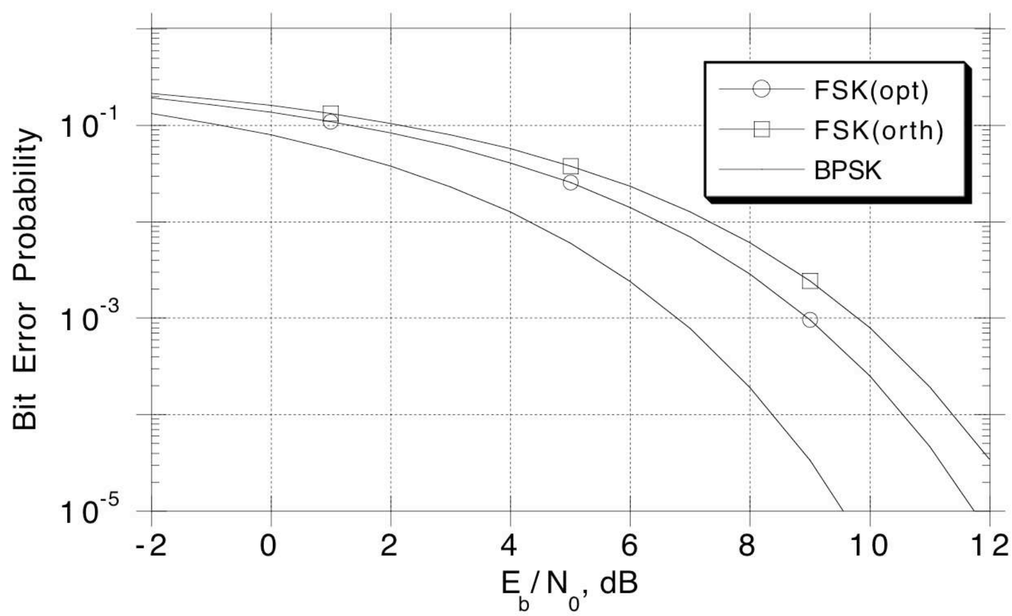
where θ is a design parameter.

Note that $E_0 = E_1 = E_b$ and that

$$\rho_{10} = \exp(-j\theta)$$

so $\mathrm{Re}\,\rho_{10}$ is minimized by $\theta = \pi$; antipodal signaling.

BEP for binary modulations:



Comparison of spectral efficiencies to Shannon bound: