MIDTERM SOLUTIONS

1. (a) The likelihood ratio is

$$\begin{aligned} L(y) &= \frac{1}{2}e^{\underline{s}^T\mathbf{\Sigma}^{-1}\underline{y}-d^2/2}+\frac{1}{2}e^{-\underline{s}^T\mathbf{\Sigma}^{-1}\underline{y}-d^2/2} \\ &= e^{-d^2/2}\cosh \underline{s}^T\mathbf{\Sigma}^{-1}\underline{y}, \end{aligned}$$

which is monotone increasing in the statistic

$$T(\underline{y}) \equiv \left|\underline{s}^T\mathbf{\Sigma}^{-1}\underline{y}\right|.$$

(Here, as usual, $d^2 = \underline{s}^T\mathbf{\Sigma}^{-1}\underline{s}$.) Thus, the Neyman-Pearson test is of the form

$$\tilde{\delta}_{NP}(\underline{y}) = \begin{cases} 1 & \text{if } T(\underline{y}) > \eta \\ 0 & \text{if } T(\underline{y}) \leq \eta\ . \end{cases}$$

(Recall that the location of equality is arbitrary.) To set the threshold η, we consider

$$\begin{aligned} P_0(T(\underline{Y}) > \eta) &= 1 - P\left(-\eta \leq \underline{s}^T\mathbf{\Sigma}^{-1}\underline{N} \leq \eta\right) \\ &= P(\underline{s}^T\mathbf{\Sigma}^{-1}\underline{N} \leq -\eta) + P(\eta \leq \underline{s}^T\mathbf{\Sigma}^{-1}\underline{N}) \\ &= \Phi(-\eta/d) + 1 - \Phi(\eta/d) \\ &= 2[1-\Phi(\eta/d)], \end{aligned}$$

where we have used the fact that $\underline{s}^T\mathbf{\Sigma}^{-1}\underline{N}$ is Gaussian with zero mean and variance d^2. Thus, the threshold for size α is

$$\eta = d\Phi^{-1}(1-\alpha/2).$$

(b) The detection probability is

$$\begin{aligned} P_D(\tilde{\delta}_{NP}) &= \frac{1}{2}P_1(T(\underline{Y}) > \eta|\Theta=+1) + \frac{1}{2}P_1(T(\underline{Y}) > \eta|\Theta=-1) \\ &= \frac{1}{2}\left[1-P\left(-\eta \leq -d^2 + \underline{s}^T\mathbf{\Sigma}^{-1}\underline{N} \leq \eta\right)\right] + \frac{1}{2}\left[1-P\left(-\eta \leq +d^2 + \underline{s}^T\mathbf{\Sigma}^{-1}\underline{N} \leq \eta\right)\right] \\ &= \frac{1}{2}\big[P(-d^2 + \underline{s}^T\mathbf{\Sigma}^{-1}\underline{N} \leq -\eta) + P(\eta \leq -d^2 + \underline{s}^T\mathbf{\Sigma}^{-1}\underline{N}) \\ &\quad + P(d^2 + \underline{s}^T\mathbf{\Sigma}^{-1}\underline{N} \leq -\eta) + P(\eta \leq d^2 + \underline{s}^T\mathbf{\Sigma}^{-1}\underline{N})\big] \\ &= \frac{1}{2}\left[\Phi\left(\frac{-\eta+d^2}{d}\right) + 1 - \Phi\left(\frac{\eta+d^2}{d}\right) + \Phi\left(\frac{-\eta-d^2}{d}\right) + 1 - \Phi\left(\frac{\eta-d^2}{d}\right)\right] \\ &= 2 - \Phi\left(\frac{\eta+d^2}{d}\right) - \Phi\left(\frac{\eta-d^2}{d}\right) \\ &= 2 - \Phi\left(\Phi^{-1}(1-\alpha/2)+d\right) - \Phi\left(\Phi^{-1}(1-\alpha/2)-d\right). \end{aligned}$$

2. (a) Here $\theta = 0$ under H_0 and $\theta = A > 0$ (known) under H_1. Thus we have the problem

$$\begin{aligned} H_0 &: \underline{Y} \sim \mathcal{N}(\underline{0}, I) \\ H_1 &: \underline{Y} \sim \mathcal{N}(\underline{0}, AD + I) \end{aligned}$$

where D is a known diagonal matrix with $D_{k,k} = s_k^2$. This is essentially a problem of detecting an independent Gaussian signal with covariance matrix $\Sigma_S = AD$ in i.i.d. Gaussian noise of variance one. We know that the decision statistic is quadratic:

$$T(\underline{y}) = \underline{y}^t Q \underline{y} \quad \text{with} \quad Q = AD(I + AD)^{-1}.$$

This can also be written

$$T(\underline{y}) = \sum_{k=1}^{n} y_k^2 \frac{As_k^2}{1 + As_k^2}.$$

The NP test has the form

$$\tilde{\delta} = \begin{cases} 1 & T(\underline{y}) \geq \eta_0 \\ 0 & T(\underline{y}) < \eta_0 \end{cases} \tag{1}$$

where η_0 is chosen to yield $P_F = \alpha$.

(b) It was claimed in the lecture/textbook that no UMP exists when

$$T(\underline{y}) = \sum_{k=1}^{n} y_k^2 \frac{\theta s_k^2}{1 + \theta s_k^2}$$

since the unknown parameter θ will not be decoupled in (**??**) after η_0 is solved to yield $P_F = \alpha$. But note that if $s_k = \pm s$ for all k, the test statistic becomes

$$T(\underline{y}) = \frac{\theta s^2}{1 + \theta s^2} \sum_{k=1}^{n} y_k^2 = \theta' \|\underline{y}\|^2 \quad \text{for } \theta' > 0$$

which leads to a rule of the form $T(\underline{y}) \gtrless \eta_0'$. Since H_0 has no dependence on θ, we know that η_0' yielding $P_F = \alpha$ will not have a dependence on θ, thus giving a UMP test. For the same reasons, a UMP test exists when $s_k \in \{0, s, -s\}$, since the zero-valued terms contribute nothing to the test statistic. To conclude, a UMP test exists when $s_k \in \{0, s, -s\}$.

(c) The LMP is formed by replacing $T(\underline{y})$ in (**??**) with the statistic

$$\left. \frac{dT(\underline{y})}{d\theta} \right|_{\theta=0} = \sum_k y_k^2 s_k^2$$

3. (a) Due to i.i.d. observations, the likelihood (parameterized by unknown b) is

$$L_b(\underline{y}) = \prod_{k=1}^{n} \frac{b}{a} e^{(a-b)y_k} = \left(\frac{b}{a}\right)^n e^{(a-b)\sum_k y_k}. \tag{2}$$

Under the assumption that $b > a$, the LMP test is a modification of the standard LRT with the likelihood replaced by

$$\left. \frac{dL_b(\underline{y})}{db} \right|_{b=a} = n\frac{b^{n-1}}{a^n} e^{(a-b)\sum_k y_k} - \frac{b^n}{a^n} \sum_k y_k e^{(a-b)\sum_k y_k} = \frac{n}{a} - \sum_k y_k.$$

(Note that if $b < a$, we could not make this claim since the derivation in the book and in class assumed $b > a$.) Thus the LRT can be expressed as

$$\left. \frac{dL_b(\underline{y})}{db} \right|_{b=a} \gtrless \eta \quad \Leftrightarrow \quad \sum_k y_k \lessgtr \frac{n}{a} + \eta = \eta'.$$

Or more formally as

$$\tilde{\delta}_{lmp}(\underline{y}) = \begin{cases} 1 & \sum_k y_k \leq \eta_0' \\ 0 & \sum_k y_k > \eta_0' \end{cases} \tag{3}$$

Note the orientation of the signs. The constant η_0 is chosen to achieve a particular false alarm rate $P_0(\Gamma_1) = \alpha$.

(b) From (**??**) and assuming $b > a$, the LRT has the form

$$L_b(\underline{y}) \underset{<}{\overset{>}{-}} \eta_0 \quad \Leftrightarrow \quad \sum_k y_k \underset{>}{\overset{<}{-}} \tfrac{1}{a-b} \log\left(\tfrac{a^n}{b^n}\eta_0\right) = \eta_0'$$

where η_0' is selected to attain $P_F = \alpha$. Since $P_F = P_0(\Gamma_1)$ is calculated from $p_0(y_k)$, it will not be a function of b, and neither will η_0' (nor possible randomization γ_0). Hence the rule above will not be a function of the unknown b, and there will exist a UMP test. Note that if a UMP test exists, it must be equivalent to the LMP test, in this case given by (**??**).

(c) We must now investigate the case $b < a$. The LMP test for this case was not covered in lecture, so instead we examine the form of the standard LRT to see if it depends on b. Taking the log of (**??**), we find

$$L_b(\underline{y}) \underset{<}{\overset{>}{-}} \eta \quad \Leftrightarrow \quad (a-b)\sum_k y_k \underset{<}{\overset{>}{-}} \log\left(\frac{a^n}{b^n}\eta\right) = \eta' \quad \Leftrightarrow \quad \begin{cases} \sum_k y_k \underset{<}{\overset{>}{-}} \frac{\eta'}{a-b} = \eta'' & a > b \\ \sum_k y_k \underset{>}{\overset{<}{-}} \frac{\eta'}{a-b} = \eta'' & a < b \end{cases}.$$

For the same reasons given in part (b), choosing η'' to yield a particular false alarm rate will yield a quantity that is independent of b. But the rule above still remains a function of b in that the rule differs depending on whether $b < a$ or $b > a$. Hence there is no UMP test.