HOMEWORK SOLUTIONS #7

1. **Poor 4.20:**

   (a) Note that $Y_1, Y_2, \ldots, Y_n$, are independent and that $Y_k \sim \mathcal{N}(0, 1+\theta s_k^2)$. Thus,

$$\frac{\partial}{\partial\theta}\log p_\theta(\underline{y}) = \sum_{k=1}^{n}\frac{\partial}{\partial\theta}\left\{-\frac{1}{2}\log(1+\theta s_k^2) - \frac{y_k^2}{2(1+\theta s_k^2)}\right\}$$

$$= -\frac{1}{2}\sum_{k=1}^{n}\left\{\frac{s_k^2}{1+\theta s_k^2} - \frac{y_k^2 s_k^2}{(1+\theta s_k^2)^2}\right\},$$

   from which the likelihood equation becomes

$$\sum_{k=1}^{n}\frac{s_k^2(y_k^2 - 1 - \hat{\theta}_{\text{ML}}(\underline{y})s_k^2)}{(1+\hat{\theta}_{\text{ML}}(\underline{y})s_k^2)^2} = 0.$$

   (b)

$$I_\theta = -E_\theta\left\{\frac{\partial^2}{\partial\theta^2}\log p_\theta(\underline{Y})\right\} = \sum_{k=1}^{n}\left\{\frac{s_k^4 E_\theta\{Y_k^2\}}{(1+\theta s_k^2)^3} - \frac{s_k^4}{2(1+\theta_k^2)^2}\right\}$$

$$= \frac{1}{2}\sum_{k=1}^{n}\frac{s_k^4}{(1+\theta s_k^2)^2}.$$

   So the CRLB is

$$\frac{2}{\sum_{k=1}^{n}\frac{s_k^4}{(1+\theta s_k^2)^2}}.$$

   (c) With $s_k^2 = 1$, the likelihood equation yields the solution

$$\hat{\theta}(\underline{y}) = \left(\frac{1}{n}\sum_{k=1}^{n} y_k^2\right) - 1,$$

   which is seen to yield a maximum of the likelihood function.

   (d) We have

$$E_\theta\left\{\hat{\theta}_{\text{ML}}(\underline{Y})\right\} = \left(\frac{1}{n}\sum_{k=1}^{n} E_\theta\left\{Y_k^2\right\}\right) - 1 = \theta.$$

   Similarly, since the $Y_k' s$ are independent,

$$Var_\theta\left(\hat{\theta}_{\text{ML}}(\underline{Y}\right) = \frac{1}{n^2}\sum_{k=1}^{n} Var_\theta\left(Y_k^2\right) = \frac{1}{n^2}\sum_{k=1}^{n} 2(1+\theta)^2 = \frac{2(1+\theta)^2}{n}.$$

   Thus, the MLE is unbiased and the variance of the MLE equals the CRLB. (Hence, the MLE is an MVUE in this case.)

2. **Poor 4.21:**
Recall that

$$p_\lambda(\underline{y}) \;=\; p_\lambda(y_1)p_\lambda(y_2) \;=\; \frac{\lambda^{(y_1+y_2)}e^{-2\lambda}}{y_1!y_2!} I_{\{y_1\geq 0,\ y_2\geq 0\}}$$

and that $\theta = e^{-\lambda}$.

(d) To find the maximum likelihood estimator of θ,

$$\begin{aligned}
\frac{\partial}{\partial\theta}\left(\log p_\lambda(\underline{y})\right) &= \frac{\partial}{\partial\theta}\left((y_1+y_2)\log\lambda - 2\lambda - \log(y_1!\ y_2!)\right) \\
&= \frac{y_1+y_2}{\lambda}\frac{\partial\lambda}{\partial\theta} - 2\frac{\partial\lambda}{\partial\theta} \\
\frac{\partial\lambda}{\partial\theta} &= -\frac{1}{\theta}
\end{aligned}$$

Setting the derivative equal to zero,

$$\begin{aligned}
\left[\frac{y_1+y_2}{-\log\theta} - 2\right]\left(\frac{-1}{\theta}\right)\bigg|_{\theta=\hat{\theta}_{\text{ML}}} &= 0 \\
\hat{\theta}_{\text{ML}}(\underline{y}) &= e^{-\frac{y_1+y_2}{2}} \\
\mathbf{E}_\lambda\left\{\hat{\theta}_{\text{ML}}(\underline{Y})\right\} &= \mathbf{E}_\lambda\left\{e^{-\frac{T}{2}}\right\} \text{ for } T\sim \text{ poisson}(2\lambda) \\
&= \sum_{t=0}^{\infty} e^{-\frac{t}{2}}\frac{(2\lambda)^t e^{-2\lambda}}{t!} = e^{-2\lambda}\sum_{t=0}^{\infty}\frac{(2\lambda/\sqrt{e})^t}{t!} = e^{-2\lambda}e^{\frac{2\lambda}{\sqrt{e}}} \\
&= e^{-\lambda\cdot 2\left(1-\frac{1}{\sqrt{e}}\right)} \ \textbf{biased} \\
\text{note that } 2\left(1-\frac{1}{\sqrt{e}}\right) &\approx 0.7869
\end{aligned}$$

However, if we create the MLE for λ directly,

$$\begin{aligned}
\frac{\partial}{\partial\lambda}\log p_\lambda(\underline{y}) &= \frac{y_1+y_2}{\lambda} - 2 \\
\Rightarrow\ \hat{\lambda}_{\text{ML}}(\underline{y}) &= \frac{y_1+y_2}{2} \\
\mathbf{E}_\lambda\left\{\hat{\lambda}_{\text{ML}}(\underline{Y})\right\} &= \lambda\ \ \textbf{unbiased}
\end{aligned}$$

(e) To determine the CRLB, we first calculate Fisher's information:

$$\begin{aligned}
I_\theta &= \mathbf{E}_\theta\left\{-\frac{\partial^2}{\partial\theta^2}\log p_\theta(\underline{Y})\right\} \\
&= \mathbf{E}_\theta\left\{\frac{1}{\theta^2}\left[\frac{Y_1+Y_2}{\log\theta} + 2 + \frac{Y_1+Y_2}{(\log\theta)^2}\right]\right\} \\
\mathbf{E}_\theta\left\{Y_1+Y_2\right\} = 2\lambda = -2\log\theta & \\
\Rightarrow\ I_\theta &= -\frac{2}{\theta^2\log\theta}
\end{aligned}$$

Finally, the CRLB for the variance of unbiased estimators of θ is $1/I_\theta$.

3. **Poor 4.23**

(a) The log-likelihood is

$$\log p(\underline{y}|A,\phi) = -\frac{1}{2\sigma^2}\sum_{k=1}^{n}\left[y_k - A\sin(\frac{k\pi}{2}+\phi)\right]^2 - \frac{n}{2}\log(2\pi\sigma^2)$$

and, differentiating with respect to the unknown parameters A and ϕ, we obtain the likelihood equations

$$\begin{aligned} \sum_{k=1}^{n}\left[y_k - \hat{A}\sin(\frac{k\pi}{2}+\hat{\phi})\right]\sin(\frac{k\pi}{2}+\hat{\phi}) &= 0 \\ \hat{A}\sum_{k=1}^{n}\left[y_k - \hat{A}\sin(\frac{k\pi}{2}+\hat{\phi})\right]\cos(\frac{k\pi}{2}+\hat{\phi}) &= 0. \end{aligned}$$

Using the identities $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ and $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, and defining the quantities

$$\begin{aligned} y_s &= \frac{2}{n}\sum_{k=1}^{n} y_k \sin(\frac{k\pi}{2}) \\ y_c &= \frac{2}{n}\sum_{k=1}^{n} y_k \cos(\frac{k\pi}{2}), \end{aligned}$$

we find that the likelihood equations can be rewritten as the pair

$$\begin{aligned} \hat{A} &= y_s\cos(\hat{\phi}) + y_c\sin(\hat{\phi}) \\ 0 &= y_c\cos(\hat{\phi}) - y_s\sin(\hat{\phi}) \end{aligned}$$

For this last step we took advantage of the facts that, for even n,

$$\begin{aligned} \frac{n}{2} &= \sum_{k=1}^{n}\sin^2(\frac{k\pi}{2}) = \sum_{k=1}^{n}\cos^2(\frac{k\pi}{2}) \\ 0 &= \sum_{k=1}^{n}\sin(\frac{k\pi}{2})\cos(\frac{k\pi}{2}) \end{aligned}$$

Putting the likelihood equations together we find

$$\begin{aligned} \hat{A}^2 &= y_s^2\cos^2(\hat{\phi}) + y_c^2\sin^2(\hat{\phi}) + 2y_s y_c\cos(\hat{\phi})\sin(\hat{\phi}) \\ &= y_s^2\cos^2(\hat{\phi}) + y_c^2\sin^2(\hat{\phi}) + y_s^2\sin^2(\hat{\phi}) + y_c^2\cos^2(\hat{\phi}) \\ &= y_s^2 + y_c^2 \end{aligned}$$

Thus,

$$\begin{aligned} \hat{A} &= \sqrt{y_s^2 + y_c^2} \\ \hat{\phi} &= \tan^{-1}(\frac{y_c}{y_s}) \end{aligned}$$

(b) The joint MAP estimator of $[A, \phi]$ solves

$$\begin{aligned}
\{\hat{A}_{\text{MAP}}, \hat{\phi}_{\text{MAP}}\} &= \arg\max_{a,\phi} w(a, \phi | \underline{y}) \\
&= \arg\max_{a,\phi} \log w(a, \phi | \underline{y}) \\
&= \arg\max_{a,\phi} \log \frac{p(\underline{y}|a, \phi) w_A(a) w_\Phi(\phi)}{p(\underline{y})} \\
&= \arg\max_{a,\phi} \log p(\underline{y}|a, \phi) + \log w_A(a) + \log w_\Phi(\phi) \\
&= \arg\max_{a,\ \phi \in [-\pi, \pi)} \log p(\underline{y}|a, \phi) + \log w_A(a)
\end{aligned}$$

We now search for the maximum of $\log p(\underline{y}|a, \phi) + \log w_A(a)$ by setting the gradient with respect to $[a, \phi]$ to zero. Similar to before, we get

$$\begin{aligned}
\sum_{k=1}^{n} \left[y_k - \hat{A} \sin(\frac{k\pi}{2} + \hat{\phi}) \right] \sin(\frac{k\pi}{2} + \hat{\phi}) + \frac{\sigma^2}{\hat{A}} - \frac{\hat{A}\sigma^2}{\beta^2} &= 0 \\
\hat{A} \sum_{k=1}^{n} \left[y_k - \hat{A} \sin(\frac{k\pi}{2} + \hat{\phi}) \right] \cos(\frac{k\pi}{2} + \hat{\phi}) &= 0.
\end{aligned}$$

Using the previously mentioned trig identities and definitions of y_s and y_c, we can obtain the pair of equations

$$\begin{aligned}
\hat{A}(1 + \alpha) - \frac{2\sigma^2}{n\hat{A}} &= y_s \cos(\hat{\phi}) + y_c \sin(\hat{\phi}) \\
0 &= y_c \cos(\hat{\phi}) - y_s \sin(\hat{\phi})
\end{aligned}$$

for $\alpha = \frac{2\sigma^2}{n\beta^2}$. Putting the previous equations together (as before) we find that

$$\hat{A}(1 + \alpha) - \frac{2\sigma^2}{n\hat{A}} = \underbrace{\sqrt{y_s^2 + y_c^2}}_{\hat{A}_{\text{ML}}}$$

and a simple application of the quadratic equation yields

$$\hat{A}_{\text{MAP}} = \frac{\hat{A}_{\text{ML}} + \sqrt{\hat{A}_{\text{ML}}^2 + \frac{8\sigma^2(1+\alpha)}{n}}}{2(1 + \alpha)}$$

It can be seen quite easily that $\hat{\phi}_{\text{MAP}} = \hat{\phi}_{\text{ML}}$.

(c) Note that, when $\beta \to \infty$, the MAP estimate of A does not approach the ML estimate of A. However, as $n \to \infty$, the MAP estimate does approach the ML estimate.