HOMEWORK SOLUTIONS #6

1. **Poor 4.12**

   The joint density of the n observations is,

$$
\begin{aligned}
p_\theta(\underline{y}) &= \begin{cases} (\theta-1)^n \prod_{k=1}^n y_k^{-\theta} & y_k \geq 1 \ \ \forall\, k \\ 0 & \text{else} \end{cases} \\
&= \begin{cases} (\theta-1)^n \exp\left(-\theta \sum_{k=1}^n \log y_k\right) & y_k \geq 1 \ \ \forall\, k \\ 0 & \text{else} \end{cases} \\
\text{let } C(\theta) &= (\theta-1)^n \\
Q(\theta) &= -\theta \\
T(\underline{y}) &= \sum_{k=1}^n \log y_k \\
h(\underline{y}) &= 1
\end{aligned}
$$

   Thus, it is clear that $p_\theta(\underline{y})$ is drawn from an exponential family. Using the Completeness Theorem for Exponential Families, we know that $T(\underline{y}) = \sum_{k=1}^n \log y_k$ is a complete sufficient statistic.

2. **Poor 4.21:**

   (a) First we write out the pdf's of our two observations:

$$
p_\lambda(y_i) = \begin{cases} \frac{\lambda^{y_i} e^{-\lambda}}{y_i!} & y_i \in \{0,1,2,3,\ldots\} \\ 0 & \text{else} \end{cases} \qquad \text{for } i = 1,2
$$

   The two observations are independent and thus,

$$
p_\lambda(\underline{y}) = p_\lambda(y_1)p_\lambda(y_2) = \frac{\lambda^{(y_1+y_2)}e^{-2\lambda}}{y_1!y_2!} I_{\{y_1\geq 0,\ y_2 \geq 0\}}
$$

   Using the factorization for an exponential family, let $\theta' = \log\lambda$

$$
\begin{aligned}
p_\lambda(\underline{y}) &= e^{\theta'(y_1+y_2)} \frac{e^{-2e^{\theta'}}}{y_1!y_2!} \\
\Rightarrow T(\underline{y}) &= y_1 + y_2 \\
h(y) &= \frac{1}{y_1!y_2!} I_{\{y_1\geq 0,\ y_2\geq 0\}} \\
C(\theta') &= e^{-2e^{\theta'}}
\end{aligned}
$$

   Clearly $T(\underline{y}) = y_1 + y_2$ is a complete sufficient statistic for θ'. Since θ and θ' are in one-to-one correspondence, sufficient for θ' implies sufficient for θ.

(b) Recall that we want to estimate $\theta = e^{-\lambda}$.

$$\begin{aligned}
\mathbf{E}_\theta\left\{\hat{\theta}(\underline{Y})\right\} &= \frac{1}{2}\mathbf{E}_\theta\left\{f(Y_1)+f(Y_2)\right\} \\
&= \frac{1}{2}\sum_{y_1=0}^{\infty} f(y_1)\frac{\lambda^{y_1}e^{-\lambda}}{y_1!} + \frac{1}{2}\sum_{y_2=0}^{\infty} f(y_2)\frac{\lambda^{y_2}e^{-\lambda}}{y_2!} \\
&= \frac{1}{2}\frac{\lambda^0 e^{-\lambda}}{0!} + \frac{1}{2}\frac{\lambda^0 e^{-\lambda}}{0!} \\
&= \frac{1}{2}e^{-\lambda} + \frac{1}{2}e^{-\lambda} \;=\; \theta
\end{aligned}$$

Thus the estimator is **unbiased**.

(c) Using the Rao-Blackwell Theorem,

$$\tilde{g}(\underline{y}) \;=\; \mathbf{E}_\theta\left\{\frac{1}{2}\left[f(Y_1)+f(Y_2)\right]\Big| T(\underline{Y}) = y_1 + y_2\right\}$$

In order to determine this MVUE we need the conditional pdfs $p_\lambda(y_i|T=t)$ for $i=1,2$. Using Bayes rule,

$$\begin{aligned}
p_\lambda(y_1|t) &= \frac{q_\lambda(t|y_1)p_\lambda(y_1)}{q_\lambda(t)} \;=\; \frac{\Pr_\lambda\{Y_2 = t - y_1\}p_\lambda(y_1)}{q_\lambda(t)} \\
&= \frac{\lambda^{(t-y_1)}e^{-\lambda}}{(t-y_1)!} \times \frac{\lambda^{(y_1)}e^{-\lambda}}{(y_1)!} \times \frac{t!}{(2\lambda)^t e^{-2\lambda}} \\
&= \left(\frac{1}{2}\right)^t \frac{t!}{(t-y_1)!\,y_1!} \\
\Rightarrow\ \mathbf{E}_\lambda\left\{f(Y_1)|t\right\} &= \sum_{y_1=0}^{\infty} f(y_1)p_\lambda(y_1|t) = \left(\frac{1}{2}\right)^t \frac{t!}{(t-0)!\,0!} = \left(\frac{1}{2}\right)^t \\
\mathbf{E}_\lambda\left\{f(Y_2)|t\right\} &= \left(\frac{1}{2}\right)^t \text{ by same method.} \\
\Rightarrow\ \tilde{g}(T(\underline{y})) &= \left(\frac{1}{2}\right)^{(y_1+y_2)}
\end{aligned}$$

By the Rao-Blackwell theorem we know that $\tilde{g}$ is unbiased, and since $\tilde{g}$ is a function of a complete sufficient statistic we know that it must be MVUE. Verifying the unbiasedness (for fun):

$$\begin{aligned}
\mathbf{E}_\lambda\left\{\tilde{g}(T)\right\} &= \sum_{t=0}^{\infty}\frac{(2\lambda)^t e^{-2\lambda}}{t!}\left(\frac{1}{2}\right)^t \\
&= e^{-2\lambda}\sum_{t=0}^{\infty}\frac{\lambda^t}{t!} = e^{-2\lambda}e^{\lambda} = e^{-\lambda} = \theta
\end{aligned}$$

3. **Uniform I.I.D. MVUE problem:**

(a)

$$p_\theta(\underline{x}) \;=\; \theta^{-n}\prod_{k=1}^{n} I_{\{x_k\in[0,\theta]\}} \;=\; \underbrace{\theta^{-n} I_{\{(\max(\underline{x})\le\theta\}}}_{g_\theta(T(\underline{x}))}\underbrace{I_{\{\min(\underline{x})\ge 0\}}}_{h(\underline{x})}$$

Thus, by the Neyman-Fisher factorization theorem, $T(\underline{x}) = \max(\underline{x})$ is a sufficient statistic.

(b) We determine the cumulative distribution function, then differentiate:

$$\begin{aligned}
\Pr_\theta[T \le t] &= \Pr_\theta[X_1, \cdots, X_n \le t] \\
&= \prod_{i=1}^{n} \Pr_\theta[X_i \le t] \\
&= \begin{cases} 0 & t < 0 \\ \left(\frac{t}{\theta}\right)^n & t \in [0,\theta] \\ 1 & t > \theta \end{cases} \\
\Rightarrow \quad q_\theta(t) &= \frac{d}{dt} \Pr_\theta[T \le t] \\
&= \begin{cases} \frac{nt^{n-1}}{\theta^n} & t \in [0,\theta] \\ 0 & \text{else} \end{cases}
\end{aligned}$$

(c) Would like to show that $\mathbf{E}\{f(T)\} = 0 \ \ \forall \theta > 0 \Rightarrow f(T) = 0$ w.p. 1. Say that

$$\mathbf{E}\{f(T)\} \; = \; \frac{n}{\theta^n} \int_0^\theta f(t) t^{n-1} dt \; = \; 0 \ \ \forall \theta > 0$$

Then

$$\begin{aligned}
\frac{\partial}{\partial \theta} \left\{ \frac{n}{\theta^n} \int_0^\theta f(t) t^{n-1} dt \right\} &= \frac{n f(\theta) \theta^{n-1}}{\theta^n} \; = \; 0 \ \ \forall \theta > 0 \\
\Rightarrow \quad f(\theta) &= 0 \ \ \forall \theta > 0 \\
\Rightarrow \quad f(T) &= 0 \ \text{ w.p. 1}
\end{aligned}$$

Hence, by definition, the sufficient statistic is complete.

(d)

$$\mathbf{E}_\theta\{T\} \; = \; \int_0^\theta \frac{nt^n}{\theta^n} dt = \frac{n}{n+1}\theta$$

(e) The MVUE is then $\hat{\theta} = \frac{n+1}{n} \max(\underline{x})$ since the estimator is unbiased and a function of the complete sufficient statistic.

(f)

$$\begin{aligned}
\mathbf{Var}_\theta(\hat{\theta}_{\text{MVUE}}) &= \mathbf{E}_\theta\{\hat{\theta}^2_{\text{MVUE}}\} - \theta^2 \\
&= \int_0^\theta \left(\frac{n+1}{n}\right)^2 t^2 \, \frac{nt^{n-1}}{\theta^n} \, d\theta \\
&= \frac{\theta^2}{n(n+2)}
\end{aligned}$$

4. **Computer Exercise**

To determine the (theoretical) variance of the alternative unbiased estimator, notice that for i.i.d. $\{X_k\}_{k=1}^n \sim \mathcal{U}[0,\theta]$,

$$\begin{aligned}
\mathbf{Var}_\theta\left(\frac{1}{n}\sum_{k=1}^n X_k\right) &= \frac{1}{n^2}\mathbf{E}\left\{\sum_i \sum_k X_i X_k\right\} - \mathbf{E}^2\left\{\frac{1}{n}\sum_k X_k\right\} \\
&= \frac{1}{n^2}\left(\sum_i \underbrace{E\{X_i^2\}}_{\theta^2/3} + \sum_i \sum_{k \ne i} \underbrace{E\{X_i\}}_{\theta/2}\underbrace{E\{X_k\}}_{\theta/2}\right) - \left(\frac{\theta}{2}\right)^2 \; = \; \frac{\theta^2}{12n}
\end{aligned}$$

Thus $\mathbf{Var}_\theta(\hat{\theta}_{\text{UE}}) = \theta^2/3n$.

Running 50 experiments with $\theta = 4.3$ for each value of n between 1 and 500, the mean and variance of the two estimators were empirically generated. The plots below demonstrate the findings, which concur with the theory above: the MVUE and UE appear to be unbiased, and their variances appear to track the theoretical quantities derived earlier. The MVUE is clearly superior to the UE in terms of variance (and thus MSE).