HOMEWORK SOLUTIONS #5

1. **Poor 4.6**

As each estimate relies on the knowledge of the conditional pdf $w(\theta|y)$, we'll determine this first.

$$\begin{aligned}
\underline{Y} &= \frac{1}{\sqrt{\Theta}}\underline{N} \\
\underline{N} &\sim \mathcal{N}(\underline{0},\Sigma) \quad \text{where } \Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \\
\underline{Y}|\theta &\sim \mathcal{N}\left(\underline{0}, \frac{1}{\theta}\Sigma\right) \\
w(\theta|\underline{y}) &= \frac{p_\theta(\underline{y})w(\theta)}{p(\underline{y})} = \frac{\frac{1}{\alpha}\frac{1}{2\pi|\frac{1}{\theta}\Sigma|^{\frac{1}{2}}}\exp\left\{-\frac{1}{2}\underline{y}^T\left(\frac{1}{\theta}\Sigma\right)^{-1}\underline{y}\right\} I_{[0,\alpha]}(\theta)}{\int_0^\alpha \frac{1}{\alpha}\frac{1}{2\pi\left|\frac{1}{\theta}\Sigma\right|^{\frac{1}{2}}}\exp\left\{-\frac{1}{2}\underline{y}^T\left(\frac{1}{\theta}\Sigma\right)^{-1}\underline{y}\right\}d\theta}
\end{aligned}$$

where $I_{[0,\alpha]}(\theta)$ equals 1 for $\theta \in [0,\alpha]$ and 0 otherwise.

$$\begin{aligned}
\text{let } \tau &= \frac{1}{2}\underline{y}^T\Sigma^{-1}\underline{y} \\
\text{then } w(\theta|\underline{y}) &= \frac{\theta e^{-\tau\theta}}{\int_0^\alpha \theta e^{-\tau\theta}d\theta}
\end{aligned}$$

$$\text{since } \left(\frac{1}{\theta}\Sigma\right)^{-1} = \theta\Sigma^{-1} \qquad \text{and } \left|\frac{1}{\theta}\Sigma\right|^{\frac{1}{2}} = \frac{1}{\theta}|\Sigma|^{\frac{1}{2}}$$

Using integration by parts,

$$\int_0^\alpha \theta e^{-\tau\theta}d\theta = \frac{1-e^{-\alpha\tau}(\alpha\tau+1)}{\tau^2}$$

(a) For the MMSE estimate, we find

$$\begin{aligned}
\hat{\theta}_{\mathsf{MMSE}}(y) &= \int_\Lambda \theta w(\theta|y)d\theta \\
&= \frac{\int_0^\alpha \theta^2 e^{-\tau\theta}d\theta}{\frac{1-e^{-\alpha\tau}(\alpha\tau+1)}{\tau^2}} \\
&= \frac{\frac{2-e^{-\alpha\tau}\left(2\alpha\tau+2+\alpha^2\tau^2\right)}{\tau^3}}{\frac{1-e^{-\alpha\tau}(\alpha\tau+1)}{\tau^2}} \\
&= \frac{2-e^{-\alpha\tau}\left(2+2\alpha\tau+\alpha^2\tau^2\right)}{\tau\left[1-e^{-\alpha\tau}(\alpha\tau+1)\right]}
\end{aligned}$$

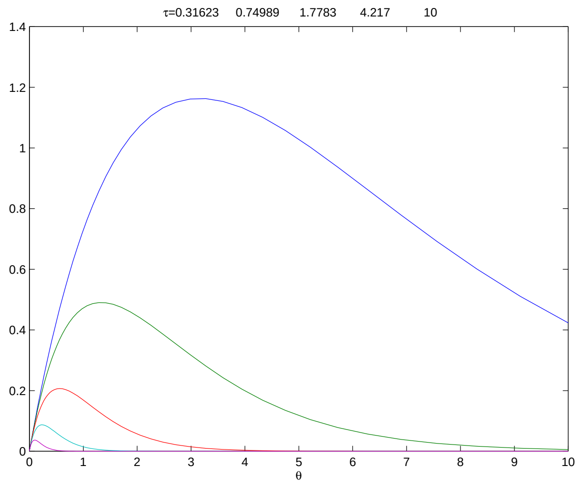
where the numerator was derived using integration by parts.

(b) For the MAP estimate,

$$\hat{\theta}_{\mathsf{MAP}}(\underline{y}) = \arg\max_\theta w(\theta|\underline{y}) = \arg\max_\theta p_\theta(\underline{y})w(\theta) = \arg\max_{\theta\in[0,\alpha]} p_\theta(\underline{y}) = \arg\max_{\theta\in[0,\alpha]} \theta e^{-\tau\theta}$$

As $\theta e^{-\tau\theta}$ is unimodal in θ (see example plots below), the maximum of is attained at the zero-derivative point.

$$\frac{\partial}{\partial\theta}\theta e^{-\theta\tau} \;=\; e^{-\theta\tau} - \tau\theta e^{-\theta\tau} \;=\; 0 \quad\Rightarrow\quad \theta \;=\; \frac{1}{\tau}$$



Thus,

$$\hat{\theta}_{\mathsf{MAP}}(\underline{y}) \;=\; \begin{cases} \alpha & \text{if } \frac{1}{\tau} > \alpha \\ \frac{1}{\tau} & \text{else} \end{cases}, \qquad \tau = \frac{\underline{y}^T\Sigma^{-1}\underline{y}}{2}$$

(c) For the MMAE estimate, we leverage the continuity of $w(\theta|y)$ to claim that

$$\hat{\theta}_{\mathsf{MMAE}} \;=\; \hat{\theta} \;\text{ s.t. }\; \frac{1}{2} = \int_0^{\hat{\theta}} w(\theta|y)d\theta = \frac{1-e^{-\hat{\theta}\tau}(\hat{\theta}\tau+1)}{1-e^{-\alpha\tau}(\alpha\tau+1)}$$

2. **Poor 4.8**

Using the indicator function

$$I_A(x) \;=\; \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

we have

$$\begin{aligned} p_N(\nu) &= e^{-\nu}I_{[0,\infty)}(\nu) \\ w(\theta) &= e^{-\theta}I_{[0,\infty)}(\theta) \end{aligned}$$

and since $Y = N + \Theta$,

$$p(y|\theta) \;=\; e^{-(y-\theta)}I_{[\theta,\infty)}(y) \;=\; e^{-(y-\theta)}I_{(-\infty,y)}(\theta) \;=\; e^{-(y-\theta)}I_{(-\infty,y)}(\theta)I_{[\theta,\infty)}(y)$$

Then

$$\begin{aligned} p(y) &= \int p(y|\theta)w(\theta)d\theta \;=\; \int_0^\infty e^{-(y-\theta)}e^{-\theta}I_{(-\infty,y)}(\theta)d\theta \\ &= \int_0^{\max(0,y)} e^{-y}d\theta \;=\; ye^{-y}I_{[0,\infty)}(y) \end{aligned}$$

Note that Y, N, Θ are positive. The posterior pdf has a uniform distribution on $[0, y)$:

$$w(\theta|y) \;=\; \frac{p(y|\theta)w(\theta)}{p(y)} \;=\; \frac{e^{-(y-\theta)}I_{(-\infty,y)}(\theta)\; e^{-\theta}I_{[0,\infty)}(\theta)}{ye^{-y}I_{[0,\infty)}(y)} \;=\; \begin{cases} \frac{1}{y} & 0 \le \theta < y \\ 0 & \text{else} \end{cases}$$

(a) The uniform distribution implies

$$\hat{\theta}_{\mathsf{MMSE}}(y) \;=\; \hat{\theta}_{\mathsf{MMAE}}(y) \;=\; \mathrm{E}\{\Theta|y\} \;=\; \frac{y}{2}$$

(b) The MMSE equals

$$\mathrm{E}\{(\hat{\theta}(Y)-\Theta)^2\} \;=\; \mathrm{E}\{\mathrm{var}(\Theta)|Y\}$$

Because $\Theta|Y \sim \mathcal{U}[0,Y)$, we know $\mathrm{var}(\Theta)|Y = Y^2/12$. Finally,

$$\mathrm{MMSE} \;=\; \mathrm{E}\{Y^2/12\} \;=\; \frac{1}{12}\int_0^\infty y^3 e^{-y} dy \;=\; \frac{1}{2}$$

(c) In this case

$$\begin{aligned} p(\underline{y}|\theta) &= \prod_{k=1}^n e^{-(y_k-\theta)} I_{[0,\infty)}(y_k) \;=\; \prod_{k=1}^n e^{-(y_k-\theta)} I_{(-\infty,y_k)}(\theta) \\ &= I_{(-\infty,\min\{y_k\})}(\theta)\; e^{n\theta} \prod_{k=1}^n e^{-y_k} \end{aligned}$$

so that

$$p(\underline{y}|\theta)w(\theta) \;=\; I_{[0,\min\{y_k\})}(\theta)\; e^{(n-1)\theta} \prod_{k=1}^n e^{-y_k}$$

Then

$$\begin{aligned} \hat{\theta}_{\mathsf{MAP}}(\underline{y}) &= \arg\max_\theta w(\theta|y) \;=\; \arg\max_\theta p(\underline{y}|\theta)w(\theta) \\ &= \arg\max_\theta I_{[0,\min\{y_k\})}(\theta)\; e^{(n-1)\theta} \\ &= \begin{cases} \{\theta \in [0,\min_k\{y_k\})\} & n=1 \\ \min_k\{y_k\} & n>1 \end{cases} \end{aligned}$$

Note that, in the case $n=1$, $\hat{\theta}_{\mathsf{MAP}}(\underline{y})$ is not uniquely specified.

3. **Poor 4.11** We can formulate this problem as

$$\underline{Y} = \Theta\underline{s} + \underline{N}$$

for

$$\begin{aligned} \underline{s} &= [\alpha, \alpha^2, \ldots, \alpha^n]^t \\ \underline{N} &\sim \mathcal{N}(\underline{0}, \sigma^2 I) \\ \Theta &\sim \mathcal{N}(0, q^2) \end{aligned}$$

This is a specific case of the model used in Example IV.B.2, “Estimation of Signal Amplitude.”

(a) From the example, we recall that $\Theta|\underline{y} \sim \mathcal{N}(m, v^2)$, where

$$\begin{aligned} m &= \frac{\sigma^{-2}\underline{s}^t\underline{y}}{\sigma^{-2}\underline{s}^t\underline{s}+q^{-2}} \;=\; \frac{\sum_{i=1}^n \alpha^i y_i}{(\sigma/q)^2 + \sum_{i=1}^n \alpha^{2i}} \\ v^2 &= \frac{1}{\sigma^{-2}\underline{s}^t\underline{s}+q^{-2}} \;=\; \frac{\sigma^2}{(\sigma/q)^2 + \sum_{i=1}^n \alpha^{2i}} \end{aligned}$$

and the conditional mean implies $\hat{\theta}_{\mathsf{MMSE}}(\underline{y}) = m$.

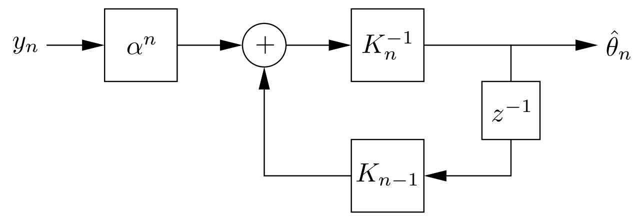
(b) The denominator of m and v^2 can be written as $K_n = K_{n-1} + \alpha^{2n}$, for $n = 1, 2, \ldots$, when $K_0 = (\sigma/q)^2$. Then we can see that

$$K_n \hat{\theta}_n(\underline{y}) \;=\; \sum_{i=1}^{n} \alpha^i y_i \;=\; \alpha^n y_n + \sum_{i=1}^{n-1} \alpha^i y_i \;=\; \alpha^n y_n + K_{n-1}\hat{\theta}_{n-1}(\underline{y})$$

where $\hat{\theta}_n$ denotes the n-sample MMSE estimator. Dividing by K_n yields the recursive formula

$$\hat{\theta}_n(\underline{y}) \;=\; K_n^{-1}\big(\alpha^n y_n + K_{n-1}\hat{\theta}_{n-1}(\underline{y})\big)$$

The block diagram is illustrated below.



(c) For the n-sample MMSE estimator, we know that the MMSE is given by the n-sample conditional variance:

$$e_n \;=\; \mathrm{E}\{(\hat{\theta}_n - \Theta)^2\} \;=\; \mathrm{E}\{\mathrm{var}(\Theta|\underline{y})\} \;=\; \mathrm{E}\{v^2\} \;=\; v^2 \;=\; \frac{\sigma^2}{(\sigma/q)^2 + \sum_{i=1}^{n} \alpha^{2i}}$$

Since $\lim_{n\to\infty} \sum_{i=1}^{n} \alpha^{2i} = \frac{1}{1-\alpha^2} - 1 = \frac{\alpha^2}{1-\alpha^2}$ when $|\alpha| < 1$,

$$\lim_{n\to\infty} e_n \;=\; \frac{\sigma^2}{(\sigma/q)^2 + \frac{\alpha^2}{1-\alpha^2}}$$

For finite n, we also note that the following cases yield perfect estimation:

$$\lim_{\sigma\to\infty} e_n \;=\; 0, \qquad \lim_{q\to 0} e_n \;=\; 0,$$