HOMEWORK SOLUTIONS #4

1. **Poor 3.14:**

(a) In this case the hypotheses are

$$\begin{aligned} H_0 &: \underline{Y} \sim \mathcal{N}(\underline{0}, I) \\ H_1 &: \underline{Y} \sim \mathcal{N}(\underline{0}, A\underline{s}\underline{s}^t + I) \end{aligned}$$

for known A, and the likelihood ratio test takes the form

$$\underline{y}^t Q \underline{y} \overset{>}{<} \eta \qquad \text{for} \qquad Q = A\underline{s}\underline{s}^t(I + A\underline{s}\underline{s}^t)^{-1}.$$

Note that Q is a rank-one matrix because it is the product of a rank-one matrix $\underline{s}\underline{s}^t$ and a full-rank matrix. Since a rank-one matrix can be written as a vector outer product, say, $Q = \underline{x}\underline{x}^t$ for appropriate $\underline{x}$, the LRT takes the form $(\underline{y}^t\underline{x})^2 \overset{>}{<} \eta$ or $|\underline{y}^t\underline{x}| \overset{>}{<} \eta'$. This is convenient because we know that $\underline{y}^t\underline{x}$ is Gaussian, in which case the P_F and P_D are easy to find. These ideas can be made concrete using the matrix inversion lemma, which says

$$(F + BCD)^{-1} \;=\; F^{-1} - F^{-1}B(DF^{-1}B + C^{-1})^{-1}DF^{-1}$$

Setting $F = I,\ B = \underline{s},\ C = A$, and $D = \underline{s}^t$, we find

$$(I + A\underline{s}\underline{s}^t)^{-1} \;=\; I - \tfrac{1}{\|s\|^2 + A^{-1}}\underline{s}\underline{s}^t$$

from which it can be shown that

$$Q \;=\; A\underline{s}\underline{s}^t(I + A\underline{s}\underline{s}^t)^{-1} \;=\; \tfrac{A}{A\|s\|^2+1}\underline{s}\underline{s}^t.$$

It is now evident that the LRT can be stated

$$|\underline{y}^t\underline{s}| \overset{>}{<} \eta''.$$

To determine the NP rule, we calculate

$$P_F \;=\; \Pr\{|\underline{y}^t\underline{s}| > \eta'' \,|H_0\} \;=\; 1 - \Pr\{-\eta'' < \underline{y}^t\underline{s} < \eta'' \,|H_0\}$$

assuming $\eta'' \geq 0$. Under H_0, we know that $\underline{y}^t\underline{s} \sim \mathcal{N}(0, \|s\|^2)$, implying that $P_F = 2\Phi(\frac{-\eta''}{\|s\|})$. For $P_F = \alpha$, we then have threshold $\eta'' = -\|s\|\Phi^{-1}(\alpha/2)$. To conclude, the NP rule is

$$|\underline{y}^t\underline{s}| \overset{>}{<} -\|s\|\Phi^{-1}(\alpha/2). \tag{1}$$

(b) The NP rule for known A is not a function of A, thus (1) is UMP in the case that A is unknown (for any $\underline{s}$).

(c) Since the LMP is equivalent to the UMP when the UMP exists, (1) is LMP.

2. **Poor 3.17:**

(a) There are two ways to do this. First the hard way, where we treat the problem as "detection of a signal with random parameter in white Gaussian noise." If we condition on $\Theta = \theta$, the hypotheses are of the form

$$\begin{aligned} H_0 &: \underline{Y} \sim \mathcal{N}(\underline{0}, \sigma^2 I) \\ H_1 &: \underline{Y}|\Theta \sim \mathcal{N}(\Theta \underline{s}, \sigma^2 I) \end{aligned}$$

The (un-conditioned) likelihood ratio then has the form

$$\begin{aligned} L(\underline{y}) &= \frac{\int p_1(\underline{y}|\theta) w(\theta) d\theta}{p_0(\underline{y})} = \int \frac{p_1(\underline{y}|\theta)}{p_0(\underline{y})} w(\theta) d\theta \\ &= \int \exp\left(\frac{\|\underline{y}\|^2 - \|y - \theta \underline{s}\|^2}{2\sigma^2} \right) w(\theta) d\theta \\ &= \int \underbrace{\exp\left(\frac{2\theta \underline{y}^t \underline{s} - \theta^2 \|\underline{s}\|^2}{2\sigma^2} \right)}_{L_\theta(\underline{y})} \frac{1}{\sqrt{2\pi v}} \exp\left(-\frac{(\theta - \mu)^2}{2v^2} \right) d\theta \\ &= \frac{1}{\sqrt{2\pi v}} \int \exp\left(\frac{2\theta \underline{y}^t \underline{s} - \theta^2}{2\sigma^2} - \frac{\theta^2 - \theta\mu + \mu^2}{2v^2} \right) d\theta \end{aligned}$$

where we used $\|\underline{s}\|^2 = 1$ in the last equation.
Completing the square in the exponent,

$$\frac{2\theta \underline{y}^t \underline{s} - \theta^2}{2\sigma^2} - \frac{\theta^2 - \theta\mu + \mu^2}{2v^2} = \underbrace{-\tfrac{\sigma^2 + v^2}{2\sigma^2 v^2}}_{A} \Big(\theta - \underbrace{\left(\tfrac{\sigma^2}{\sigma^2 + v^2}\right)\left(\mu + \tfrac{v^2 \underline{s}^t \underline{y}}{\sigma^2}\right)}_{B} \Big)^2 + \underbrace{\tfrac{1}{2v^2}\left(\tfrac{\sigma^2}{\sigma^2 + v^2}\right)}_{C} \left(\mu + \tfrac{v^2 \underline{s}^t \underline{y}}{\sigma^2}\right)^2 + \underbrace{\tfrac{\mu^2}{2v^2}}_{D}$$

which implies that

$$L(\underline{y}) = \exp\left(C \left(\mu + \tfrac{v^2 \underline{s}^t \underline{y}}{\sigma^2}\right)^2 \right) \underbrace{\tfrac{\exp(D)}{\sqrt{2\pi v}} \int \exp(A(\theta - B)^2) d\theta}_{E}$$

Since the LRT has the form $L(\underline{y}) \gtrless \eta$ and since $E > 0$ and $C > 0$, an equivalent LRT is given by

$$\tfrac{1}{C} \ln\left(\tfrac{L(\underline{y})}{E} \right) = \left(\mu + \tfrac{v^2 \underline{s}^t \underline{y}}{\sigma^2}\right)^2 \gtrless \eta'$$

Now the easy way. Realize that $\underline{Y}$ is Gaussian under H_1 with mean $E\{\Theta \underline{s} + \underline{N}\} = \underline{s} E\{\Theta\} + E\{\underline{N}\} = \mu \underline{s}$ and with covariance $E\{((\Theta - \mu)\underline{s} + \underline{N})((\Theta - \mu)\underline{s} + \underline{N})^t\} = E\{(\Theta - \mu)^2\} \underline{s}\underline{s}^t + E\{\underline{N}\underline{N}^t\} = v^2 \underline{s}\underline{s}^t + \sigma^2 I$, using the uncorrelatedness of Θ and $\underline{N}$. Essentially we face the problem of "detecting a colored Gaussian signal in white Gaussian noise," where the signal has autocovariance matrix $\mathbf{\Sigma}_S = v^2 \underline{s}\underline{s}^t$. From (III.B.84), the log likelihood takes the form

$$\log L(\underline{y}) = \frac{1}{2} \underline{y}^t (\sigma^{-2} I - (\sigma^2 I + v^2 \underline{s}\underline{s}^t)^{-1}) \underline{y} + \mu \underline{s}^t (\sigma^2 I + v^2 \underline{s}\underline{s}^t)^{-1} \underline{y} + C,$$

Being clever, we use either the Matrix Inversion Lemma (found in many linear algebra or signal processing books) or a carefully constructed eigendecomposition to find that

$$(\sigma^2 I + v^2 \underline{s}\underline{s}^t)^{-1} = \frac{1}{\sigma^2} I - \frac{v^2}{(v^2 + \sigma^2)\sigma^2} \underline{s}\underline{s}^t.$$

Plugging the previous expression into the log likelihood ratio, we find a detector with the desired form.

(b) False alarm probability is

$$\begin{aligned} P_F &= \Pr\left\{ |\mu + \tfrac{v^2}{\sigma^2}\underline{s}^t\underline{Y}| > \sqrt{\eta'} \mid H_0 \right\} \\ &= \Pr\left\{ \mu + \tfrac{v^2}{\sigma^2}\underline{s}^t\underline{Y} > \sqrt{\eta'} \mid H_0 \right\} + \Pr\left\{ \mu + \tfrac{v^2}{\sigma^2}\underline{s}^t\underline{Y} < -\sqrt{\eta'} \mid H_0 \right\} \\ &= \Pr\left\{ \underline{s}^t\underline{Y} > \tfrac{\sqrt{\eta'}-\mu}{v^2/\sigma^2} \mid H_0 \right\} + \Pr\left\{ \underline{s}^t\underline{Y} < \tfrac{-\sqrt{\eta'}-\mu}{v^2/\sigma^2} \mid H_0 \right\} \end{aligned}$$

Under H_0 we know $\underline{s}^t\underline{Y} \sim \mathcal{N}(0, \|\underline{s}\|^2\sigma^2)$ where $\|\underline{s}\|^2 = 1$, thus, under H_0,

$$\underline{s}^t\underline{Y} \;\sim\; \mathcal{N}(0, \sigma^2)$$

and so

$$P_F \;=\; 1 - \Phi\Big(\frac{\sqrt{\eta'} - \mu}{v^2/\sigma}\Big) + \Phi\Big(\frac{-\sqrt{\eta'} - \mu}{v^2/\sigma}\Big)$$

Using the same arguments, detection probability is

$$P_D \;=\; \Pr\left\{ \underline{s}^t\underline{Y} > \tfrac{\sqrt{\eta'}-\mu}{v^2/\sigma^2} \mid H_1 \right\} + \Pr\left\{ \underline{s}^t\underline{Y} < \tfrac{-\sqrt{\eta'}-\mu}{v^2/\sigma^2} \mid H_1 \right\}$$

Under H_1 we know $\underline{s}^t\underline{Y} = \Theta + \underline{s}^t\underline{N} \sim \mathcal{N}(\mu, v^2 + \sigma^2)$, leveraging the fact that $\|\underline{s}\|^2 = 1$. So,

$$P_D \;=\; 1 - \Phi\left(\frac{\frac{\sqrt{\eta'}-\mu}{v^2/\sigma^2} - \mu}{\sqrt{v^2+\sigma^2}} \right) + \Phi\left(\frac{\frac{-\sqrt{\eta'}-\mu}{v^2/\sigma^2} - \mu}{\sqrt{v^2+\sigma^2}} \right)$$

3. **Poor 3.20**

We note that, $P_e = \pi_0 P_F + \pi_1 P_M$. Since we have a Bayesian problem, we do not need to consider randomization and we define $\Gamma_1 = \{y : \log L(y) \geq \tau\}$ and $\Gamma_0 = \Gamma_1^c = \{y : \log L(y) < \tau\}$. (Note the typographical error in the text.) Then,

$$P_e \;=\; \pi_0 \int_{\Gamma_1} p_0(y)dy + \pi_1 \int_{\Gamma_0} p_1(y)dy$$

For any $s \geq 0$

$$\begin{aligned} \Gamma_1 &= \{y : e^{s \log L(y)} \geq e^{s\tau}\} \\ &= \{y : L^s(y) \geq e^{s\tau}\} \\ &= \{y : L^s(y) e^{-s\tau} \geq 1\} \end{aligned}$$

Similarly, for any $t \leq 0$

$$\begin{aligned} \Gamma_0 &= \{y : e^{t \log L(y)} \geq e^{t\tau}\} \\ &= \{y : L^t(y) \geq e^{t\tau}\} \\ &= \{y : L^t(y) e^{-t\tau} \geq 1\} \end{aligned}$$

Then, for $t \leq 0$ and $s \geq 0$,

$$\begin{aligned} P_e &= \pi_0 \int_{\Gamma_1} p_0(y)dy + \pi_1 \int_{\Gamma_0} p_1(y)dy \\ &\leq \pi_0 \int_{\Gamma_1} e^{-s\tau} L^s(y) p_0(y)dy + \pi_1 \int_{\Gamma_0} e^{-t\tau} L^t(y) p_1(y)dy \\ &= \pi_0 e^{-s\tau} \int_{\Gamma_1} L^s(y) p_0(y)dy + \pi_1 e^{-t\tau} \int_{\Gamma_0} L^{(t+1)}(y) p_0(y)dy \end{aligned}$$

Thus for $0 \leq s \leq 1$

$$P_e \leq \pi_0 e^{-s\tau} \int_{\Gamma_1} L^s(y) p_0(y) dy + \pi_1 e^{(1-s)\tau} \int_{\Gamma_0} L^s(y) p_0(y) dy$$

4. **Poor 3.21**

From (II.B.24) we know that the minimum probability of error is $\min\{\lambda, 1-\lambda\}$. Using (III.C.18) we have, for the equal prior case,

$$P_e \leq \left(\frac{1}{2}\right)^{1-s} \left(\frac{1}{2}\right)^{s} e^{\mu_{T,0}(s)} = \frac{1}{2}\mathbf{E_0}\left[L^s(Y)\right]$$

Since the observations live in the set $\{0, 1\}$,

$$\begin{aligned} \mathbf{E_0}\left[L^s(Y)\right] &= \sum_{y=0}^{1} \left(\frac{p_1(y)}{p_0(y)}\right)^s p_0(y) = \sum_{y=0}^{1} (p_0(y))^{1-s}(p_1(y))^s \\ &= (1-\lambda)^{1-s}\lambda^s + \lambda^{1-s}(1-\lambda)^s \end{aligned}$$

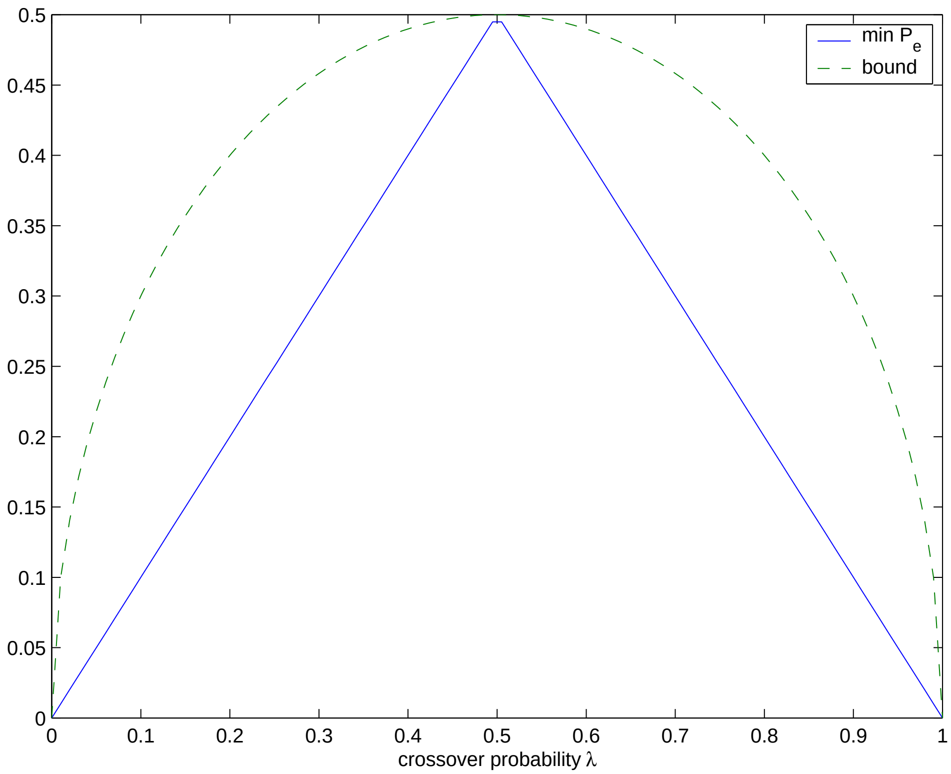
This quantity is minimized for some $s \in (0, 1)$. By symmetry, we see that $s_0 = \frac{1}{2}$.

$$\min_{s \in (0,1)} \mathbf{E_0}\left[L^s(Y)\right] = 2\sqrt{\lambda(1-\lambda)}$$

The Chernoff bound is

$$P_e \leq \frac{1}{2}\mathbf{E_0}\left[L^s(Y)\right] = \sqrt{\lambda(1-\lambda)}$$

which happens to be the Bhattacharya bound (III.C.22).


min P_e
bound
crossover probability λ
0
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1