HOMEWORK SOLUTIONS #3

1. **Titanic experiment.** My code generated the values

	y_η	P_D
empirical	1.091	0.795
theoretical	1.046	0.831

assuming the following NP rule for $P_F \leq 0.3$.

$$\tilde{\delta}(y) = \begin{cases} 1 & |y| \geq y_\eta \\ 0 & |y| < y_\eta \end{cases}.$$

The strategy I used was the following. With the assuming that the likelihood test can be equivalently stated as

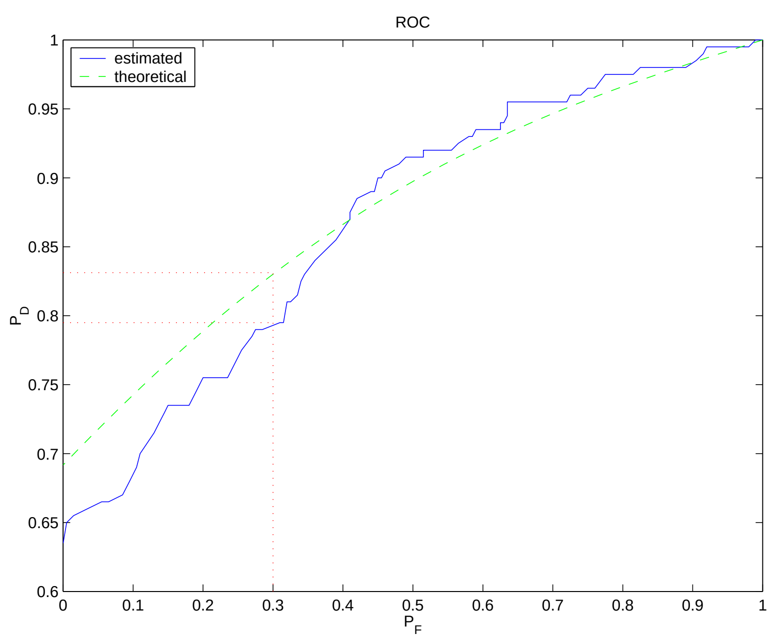
$$L(y) \geq \eta \quad \Leftrightarrow \quad |y| \geq y_\eta,$$

I created a vector of thresholds $\underline{y_\eta}$ and estimated P_F and P_D for each element in $\underline{y_\eta}$ as

$$P_F = \Pr\{|y| \geq y_\eta | H_0\} \approx \frac{\text{\# data points in column 1 with absolute value greater than } y_\eta}{200}.$$

$$P_D = \Pr\{|y| \geq y_\eta | H_1\} \approx \frac{\text{\# data points in column 2 with absolute value greater than } y_\eta}{200}.$$

then I plotted the resulting vectors $\underline{P_D}$ versus $\underline{P_F}$. From this plot, y_{η_0} was selected as the threshold closest to $P_F = \alpha = 0.3$.



2. **Poor 2.20:**
The likelihood ratio is given by

$$L(y) = \frac{\prod_{k=1}^{n} \frac{1}{\sqrt{2\pi}\sigma_1} e^{-(y_k-\mu_1)^2/2\sigma_1^2}}{\prod_{k=1}^{n} \frac{1}{\sqrt{2\pi}\sigma_0} e^{-(y_k-\mu_0)^2/2\sigma_0^2}}$$
$$= \left(\frac{\sigma_0}{\sigma_1}\right)^n e^{\frac{n}{2}\left(\frac{\mu_0^2}{\sigma_0^2}-\frac{\mu_1^2}{\sigma_1^2}\right)} e^{\left(\frac{1}{2\sigma_0^2}-\frac{1}{2\sigma_1^2}\right)\sum_{k=1}^{n} y_k^2} e^{\left(\frac{\mu_1}{\sigma_1^2}-\frac{\mu_0}{\sigma_0^2}\right)\sum_{k=1}^{n} y_k},$$

which shows the structure indicated.

Manipulating this function for the special case where $\mu_0 = 0$, $\mu_1 = \mu > 0$ and $\sigma^2 = \sigma_1^2 = \sigma_2^2$,

$$L(\underline{y}) = e^{\frac{\mu^2}{\sigma^2}\left(\sum_{k=1}^{n} y_k \; - \frac{n}{2}\right)}$$

Since $\log(\cdot)$ is a monotone increasing function, an LRT of the form $L(y) \gtrless \eta$ can be manipulated into an equivalent LRT of the form

$$\sum_{k=1}^{n} y_k \gtrless y_\eta$$

with $y_\eta = \frac{\sigma^2\eta}{\mu} + \frac{\mu}{2}$. Though our rule may appear to be independent of σ, we cannot say for sure until we derive an expression for y_η in terms of false alarm rate α. Noting that y_k are i.i.d. $\sim \mathcal{N}(0, \sigma^2)$ under hypothesis H_0, their n-fold sum has distribution $\mathcal{N}(0, n\sigma^2)$. Thus

$$P_F = \alpha = \Pr[\sum_{k=1}^{n} y_k > y_\eta | H_0]$$
$$= 1 - \Phi\left(\frac{y_\eta}{\sqrt{n}\sigma}\right)$$
$$\Rightarrow y_\eta = \sqrt{n}\sigma\Phi^{-1}(1-\alpha)$$

Since the decision rule is clearly a function of σ, a UMP test does not exist.

The GLRT is formed by maximizing $p_i(\underline{y})$ with respect to σ^2. A necessary condition for the maximizer is the property $\frac{\partial}{\partial(\sigma^2)} p_i(\underline{y}) = 0$. Doing this under H_1 and H_0 we get

$$\hat{\sigma}_1^2 = \frac{1}{n}\sum_{k=1}^{n}(y_k - \mu)^2$$
$$\hat{\sigma}_0^2 = \frac{1}{n}\sum_{k=1}^{n} y_k^2$$

It can be shown that $p_i(\underline{y})$ is convex in σ^2, and thus the zero-derivative condition is both necessary and sufficient. (Alternatively, you could compute the second derivative and verify that it is negative at this zero-derivative point, implying a maximum.) If we substitute these ML variance estimates into the likelihood ratio, the resulting GLRT "likelihood ratio" is,

$$L_G(\underline{y}) = \frac{\left(\frac{n}{\sum_{k=1}^{n}(y_k-\mu)^2}\right)^{\frac{n}{2}} e^{-\frac{1}{2}}}{\left(\frac{n}{\sum_{k=1}^{n} y_k^2}\right)^{\frac{n}{2}} e^{-\frac{1}{2}}} = \left(\frac{\sum_{k=1}^{n} y_k^2}{\sum_{k=1}^{n}(y_k-\mu)^2}\right)^{\frac{n}{2}}$$

The GLRT then has the form $L_G(\underline{y}) \gtrless \eta$ for appropriately chosen threshold η.

3. **Poor 3.3**

(a) From Exercise 16 of Chapter II, the optimum test has critical regions:

$$\Gamma_k = \left\{ \underline{y} \in R^n \,\middle|\, p_k(\underline{y}) = \max_{0 \le l \le M-1} p_l(\underline{y}) \right\}.$$

Since $p_l(\underline{y})$ is the $N(\underline{s}_l, \sigma^2 \mathbf{I})$ density, this reduces to

$$\Gamma_k = \left\{ \underline{y} \in R^n \,\middle|\, \|\underline{y} - \underline{s}_k\|^2 = \min_{0 \le l \le M-1} \|y - s_l\|^2 \right\} = \left\{ \underline{y} \in R^n \,\middle|\, \underline{s}_k^T \underline{y} = \max_{0 \le l \le M-1} \underline{s}_l^T \underline{y} \right\}.$$

(b) Using P_d to denote probability of detection, the equal priors assumption yields

$$P_e = 1 - P_d = 1 - \frac{1}{M} \sum_{k=0}^{M-1} P_k(\Gamma_k)$$

where

$$P_k(\Gamma_k) = \Pr\left\{ \underline{s}_k^T \underline{y} \ge \underline{s}_0^T \underline{y},\ \underline{s}_k^T \underline{y} \ge \underline{s}_1^T \underline{y}, \dots,\ \underline{s}_k^T \underline{y} \ge \underline{s}_{M-1}^T \underline{y} \mid H_k \right\}$$

Denoting $t_i = \underline{s}_i^T \underline{y}$, and recalling that $\underline{Y} = \underline{s}_k + \underline{N}$ under H_k, we find

$$p(t_i | H_k) \sim \begin{cases} \mathcal{N}(0, \|\underline{s}\|^2 \sigma^2) & i \ne k \\ \mathcal{N}(\|\underline{s}\|^2, \|\underline{s}\|^2 \sigma^2) & i = k \end{cases}$$

and that T_i are independent (under H_k) due to the orthogonality of $\{\underline{s}_k\}$. Then

$$\begin{aligned} P_k(\Gamma_k) &= \Pr\{t_k \ge t_0,\ t_k \ge t_1, \dots,\ t_k \ge t_{M-1} \mid H_k\} \\ &= \int \Pr\{x \ge t_0,\ x \ge t_1, \dots,\ x \ge t_{M-1} \mid H_k\}\, p_{T_k}(x) dx \\ &= \int \left(\prod_{i \ne k} \Pr\{x \ge t_i \mid H_k\} \right) p_{T_k}(x) dx \\ &= \int \left[\Phi\left(\frac{x}{\|\underline{s}\|\sigma} \right) \right]^{M-1} \frac{1}{\sqrt{2\pi}\|\underline{s}\|\sigma} e^{-\frac{1}{2\|\underline{s}\|^2\sigma^2}(x - \|\underline{s}\|^2)^2} dx \\ &= \int [\Phi(y)]^{M-1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y - \frac{\|\underline{s}\|}{\sigma})^2} dy \end{aligned}$$

Noting that the right side above does not depend on k, we conclude that

$$P_e = 1 - P_0(\Gamma_0) = 1 - \frac{1}{\sqrt{2\pi}} \int [\Phi(y)]^{M-1} e^{-\frac{1}{2}(y - \frac{\|\underline{s}\|}{\sigma})^2} dy$$

4. **Poor 3.9:**

(a) Consider H_1,

$$\begin{aligned} p_1(\underline{y}|\theta=1) &\sim \mathcal{N}(A\underline{s}^{(1)}, I) \\ p_1(\underline{y}|\theta=0) &\sim \mathcal{N}(A\underline{s}^{(0)}, I) \\ p_1(\underline{y}) &= \frac{1}{2}p_1(\underline{y}|\theta=1) + \frac{1}{2}p_1(\underline{y}|\theta=0) \end{aligned}$$

Note that (as in problem 3.7) the pdf of $Y|H_1$ is a *Gaussian mixture* density, which is the sum of two Gaussian pdf's. Note that adding densities is very different from adding random variables (i.e., convolving densities)! We first form the likelihood ratio

$$\begin{aligned} L(\underline{y}) &= \frac{p(\underline{y}|H_1)}{p(\underline{y}|H_0)} = \frac{\int p(\underline{y}|H_1,\theta)w(\theta|H_1)d\theta}{\int p(\underline{y}|H_0,\theta)w(\theta|H_0)d\theta} = \frac{\frac{1}{2}p(\underline{y}|H_1,\theta=1)+\frac{1}{2}p(\underline{y}|H_1,\theta=-1)}{p(\underline{y}|H_0)} \\ &= \frac{1}{2}\exp\left(\frac{1}{2}\left[2A\underline{s}^{(1)T}\underline{y} - A^2\|\underline{s}^{(1)}\|^2\right]\right) + \frac{1}{2}\exp\left(\frac{1}{2}\left[2A\underline{s}^{(0)^T}\underline{y} - A^2\|\underline{s}^{(0)}\|^2\right]\right) \\ &\quad \text{and since } \|s^{(i)}\|^2 = 1 \\ &= \left(\frac{1}{2}\exp(-A^2/2)\right) \times \left\{\exp\left[A\underline{s}^{(1)T}\underline{y}\right] + \exp\left[A\underline{s}^{(0)T}\underline{y}\right]\right\} \end{aligned}$$

(b) If $A > 0$ is unknown, we have a composite test. We form the locally optimal likelihood ratio function,

$$\begin{aligned} \frac{d}{dA}\,L_A(\underline{y})\Big|_{A=0} &= \frac{d}{dA}\left(\frac{1}{2}\exp(-A^2/2)\right) \times \left\{\exp\left[A\underline{s}^{(1)T}\underline{y}\right] + \exp\left[A\underline{s}^{(0)T}\underline{y}\right]\right\}\Big|_{A=0} \\ &= -A\left(\exp(-A^2/2)\right) \times \left\{\exp\left[A\underline{s}^{(1)T}\underline{y}\right] + \exp\left[A\underline{s}^{(0)T}\underline{y}\right]\right\} + \\ &\quad \left(\frac{1}{2}\exp(-A^2/2)\right) \times \left\{\underline{s}^{(1)T}\underline{y}\exp\left[A\underline{s}^{(1)T}\underline{y}\right] + \underline{s}^{(0)^T}\underline{y}\exp\left[A\underline{s}^{(0)^T}\underline{y}\right]\right\}\Big|_{A=0} \\ &= \frac{1}{2}\underline{y}^T\left(\underline{s}^{(1)} + \underline{s}^{(0)}\right) \end{aligned}$$

Thus the locally optimal test has the form

$$\delta(\underline{y}) = \begin{cases} 1 & \text{if } \underline{y}^T\left(\underline{s}^{(1)} + \underline{s}^{(0)}\right) \geq \eta \\ 0 & \qquad\qquad\qquad\quad < \eta \end{cases}$$

since the density of $\underline{Y}$ is continuous, implying no need for randomization.

(c) For $A=$ *known constant*, we next determine the receiver operating characteristics for the receiver above. We consider the statistics of the test statistic under both hypotheses,

$$\begin{aligned} T(\underline{y}) &= \underline{y}^T\left(\underline{s}^{(1)} + \underline{s}^{(0)}\right) \\ T(\underline{y})|H_0 &= \underline{n}^T\left(\underline{s}^{(1)} + \underline{s}^{(0)}\right) \\ \rightarrow \quad T(\underline{Y})|H_0 &\sim \text{Gaussian} \\ \mathbf{E}\{T(\underline{Y})|H_0\} &= \mathbf{E}\{\underline{N}\}^T\left(\underline{s}^{(1)} + \underline{s}^{(0)}\right) = 0 \\ \mathbf{Var}\{T(\underline{Y})|H_0\} &= \left(\underline{s}^{(1)} + \underline{s}^{(0)}\right)^T \mathbf{E}\{\underline{NN}^T\}\left(\underline{s}^{(1)} + \underline{s}^{(0)}\right) \\ &= \|\underline{s}^{(1)}\|^2 + 2\underline{s}^{(1)^T}\underline{s}^{(0)} + \|\underline{s}^{(0)}\|^2 = 1 + 0 + 1 = 2 \\ \rightarrow \quad T(\underline{Y})|H_0 &\sim \mathcal{N}(0, 2) \end{aligned}$$

It is straightforward to determine the threshold for $P_F = \alpha$,

$$\begin{aligned} P_0[\Gamma_1] &= 1 - \Phi\left(\frac{\eta}{\sqrt{2}}\right) = \alpha \\ \to \quad \eta &= \sqrt{2}\Phi^{-1}(1-\alpha) \end{aligned}$$

Given this threshold, we next determine the probability of detection. We begin by considering the conditional statistics of $T(\underline{Y})|H_1, \Theta = \theta$:

$$\begin{aligned} T(\underline{Y})|H_1, \Theta = \theta &= \left(\underline{N} + A(1-\theta)\underline{s}^{(0)} + A\theta\underline{s}^{(1)}\right)^T \left(\underline{s}^{(1)} + \underline{s}^{(0)}\right) \\ &= \underline{N}^T \left(\underline{s}^{(1)} + \underline{s}^{(0)}\right) + A(1-\theta) + A\theta \\ &= \underline{N}^T \left(\underline{s}^{(1)} + \underline{s}^{(0)}\right) + A \\ \to \quad T(\underline{Y})|H_1, \Theta = \theta &\sim \mathcal{N}(A, 2) \end{aligned}$$

Therefore,

$$\begin{aligned} p_1(\underline{y}|\theta = 1) &= p_1(\underline{y}|\theta = 0) \\ \to \quad p_1(\underline{y}) &= \mathcal{N}(A, 2) \end{aligned}$$

Since the statistics of $T(\underline{Y})$ are Gaussian and independent of Θ, the detection probability is

$$\begin{aligned} P_D &= P_1(\Gamma_1) = 1 - \Phi\left(\frac{\eta - A}{\sqrt{2}}\right) \\ &= 1 - \Phi\left(\Phi^{-1}(1-\alpha) - \frac{A}{\sqrt{2}}\right) \end{aligned}$$

5. **Poor 3.11:**
Here we aim to calculate

$$P_e = \frac{1}{2}P_0(\Gamma_1) + \frac{1}{2}P_1(\Gamma_0) = P_0(\Gamma_1),$$

where the latter equality follows from the symmetry of the problem. From the form of the LRT (III.B.81), we know that

$$\begin{aligned} P_0(\Gamma_1) &= \Pr\{Y_{c1}^2 + Y_{s1}^2 > Y_{c0}^2 + Y_{s0}^2 \mid H_0\} \\ &= \int\int_{\mathbb{R}^2} \Pr\{Y_{c1}^2 + Y_{s1}^2 > y_{c0}^2 + y_{s0}^2 \mid H_0\} p_{Y_{c0},Y_{s0}|H_0}(y_{c0}, y_{s0}) dx, \end{aligned}$$

where, for $j \in \{1, 2\}$,

$$\begin{aligned} y_{cj} &= \textstyle\sum_{k=1}^n a_{jk} y_k \cos[(k-1)\omega_c T_S] \\ y_{sj} &= \textstyle\sum_{k=1}^n a_{jk} y_k \sin[(k-1)\omega_c T_S] \end{aligned}$$

The statistics of Y_{cj} and Y_{sj} must be investigated further. Observe that, using the identity $\sin(a)\cos(b) = \frac{1}{2}\sin(a+b) + \frac{1}{2}\sin(a-b)$ and assumption that the "double frequency sine term" equals zero for any θ, as stated previous to (III.B.82),

$$\begin{aligned} y_{cj}|H_j, \theta &= \sum_k a_{jk} \cos[(k-1)\omega_c T_S]\big(a_{jk}\sin[(k-1)\omega_c T_S + \theta] + n_k\big) \\ &= \sum_k a_{jk}^2 \left(\frac{1}{2}\sin[2(k-1)\omega_c T_S + \theta] + \frac{1}{2}\sin(\theta)\right) + \sum_k a_{jk} n_k \cos[(k-1)\omega_c T_S]\big) \\ &= \frac{1}{2}\sum_k a_{jk}^2 \sin(\theta) + \sum_k a_{jk} n_k \cos[(k-1)\omega_c T_S]\big) \\ \Rightarrow \quad & \begin{cases} \mathrm{E}\{Y_{cj} \mid H_j, \theta\} &= \frac{n\overline{a^2}}{2}\sin(\theta) \\ \mathrm{var}\{Y_{cj} \mid H_j, \theta\} &= \frac{n\overline{a^2}}{2}\sigma^2 \end{cases} \end{aligned}$$

where in the last statement we have used the assumption that the "double frequency term" (i.e., the rightmost term in (III.B.67)) equals zero. The same procedure applied to y_{sj} yields

$$\begin{aligned} y_{sj}|H_j,\theta &= \frac{1}{2}\sum_k a_{jk}^2 \cos(\theta) + \sum_k a_{jk} n_k \sin[(k-1)\omega_c T_S]) \\ &\Rightarrow \begin{cases} \mathrm{E}\{Y_{sj} \mid H_j,\theta\} &= \frac{n\overline{a^2}}{2}\cos(\theta) \\ \mathrm{var}\{Y_{sj} \mid H_j,\theta\} &= \frac{n\overline{a^2}}{2}\sigma^2 \end{cases} \end{aligned}$$

Observing that $y_{cj}|H_j,\theta \sim \mathcal{N}$, the statistical characterization of these variables is completed by

$$\begin{aligned} \mathrm{cov}\{Y_{cj}, Y_{sj} \mid H_j, \theta\} &= \mathrm{E}\left\{\sum_k a_{jk} n_k \cos[(k-1)\omega_c T_S] \sum_l a_{jl} n_l \sin[(l-1)\omega_c T_S]\right\} \\ &= \frac{\sigma_2}{2}\sum_k a_{jk}^2 \sin[2(k-1)\omega_c T_S] \;=\; 0 \end{aligned}$$

where again we use the "double frequency term" assumption. To conclude,

$$\begin{bmatrix} Y_{cj} \\ Y_{sj} \end{bmatrix}\Bigg| H_j,\theta \;\sim\; \mathcal{N}\left(\frac{n\overline{a^2}}{2}\begin{bmatrix}\sin(\theta)\\ \cos(\theta)\end{bmatrix}, \frac{n\overline{a^2}}{2}\sigma^2 I\right)$$

Repeating essentially the same steps, but incorporating the orthogonality between $\{a_{1k}\}$ and $\{a_{jk}\}$, we find that

$$\begin{bmatrix} Y_{cj} \\ Y_{sj} \end{bmatrix}\Bigg| H_{1-j},\theta \;\sim\; \mathcal{N}\left(0, \frac{n\overline{a^2}}{2}\sigma^2 I\right)$$

Note that the preceding pdfs are conditioned on both H_i and θ, while we are interested in pdfs conditioned only on H_i. Because $\left[\begin{smallmatrix} Y_{cj} \\ Y_{sj} \end{smallmatrix}\right] |H_{i-j},\theta$ has no θ dependence, we conclude that

$$\begin{bmatrix} Y_{cj} \\ Y_{sj} \end{bmatrix}\Bigg| H_{i-j} \;\sim\; \mathcal{N}\left(0, \frac{n\overline{a^2}}{2}\sigma^2 I\right)$$

For $\left[\begin{smallmatrix} Y_{cj} \\ Y_{sj} \end{smallmatrix}\right]\Big| H_j$, however, we must average over θ:

$$p_{Y_{cj},Y_{sj}|H_j}(y_{cj},y_{sj}) = \frac{1}{2\pi}\int_0^{2\pi} p_{Y_{cj},Y_{sj}|H_j,\theta}(y_{cj},y_{sj}|\theta)d\theta$$

We now can use these pdfs for performance evaluation. Repeating the argument in (III.B.72), we find that

$$\begin{aligned} \Pr\{Y_{c1}^2 + Y_{s1}^2 > x \mid H_0\} &= \int\int_{y_{c1}^2+y_{s1}^2>x} \frac{1}{\pi n\sigma_2\overline{a^2}}\, e^{-\frac{y_{c1}^2+y_{s1}^2}{n\sigma_2\overline{a^2}}}\, dy_{c1}dy_{s1} \\ &= \frac{1}{\pi n\sigma_2\overline{a^2}}\int_0^{2\pi}\int_x^{\infty} r e^{-\frac{r^2}{n\sigma_2\overline{a^2}}} dr d\psi \\ &= \exp\left(-\frac{x}{n\sigma_2\overline{a^2}}\right) \end{aligned}$$

Then, with the substitution $x = y_{c0}^2 + y_{s0}^2$,

$$\begin{aligned}
P_e &= \int\int_{\mathbb{R}^2} \exp\left(-\frac{y_{c0}^2 + y_{s0}^2}{n\sigma_2\overline{a^2}}\right) \left[\frac{1}{2\pi}\int_0^{2\pi} p_{Y_{cj},Y_{sj}|H_j,\theta}(y_{cj}, y_{sj}|\theta) d\theta\right] dy_{c0} dy_{s0} \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left[\int\int_{\mathbb{R}^2} \exp\left(-\frac{y_{c0}^2 + y_{s0}^2}{n\sigma_2\overline{a^2}}\right) p_{Y_{cj},Y_{sj}|H_j,\theta}(y_{cj}, y_{sj}|\theta) dy_{c0} dy_{s0}\right] d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left[\int\int_{\mathbb{R}^2} \exp\left(-\frac{y_{c0}^2 + y_{s0}^2}{n\sigma_2\overline{a^2}}\right) \frac{1}{\pi n\sigma_2\overline{a^2}} \exp\left(-\frac{(y_{c0} - \frac{n\overline{a^2}}{2}\sin\theta)^2 + (y_{s0} - \frac{n\overline{a^2}}{2}\cos\theta)^2}{n\sigma_2\overline{a^2}}\right) dy_{c0} dy_{s0}\right] d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left[\frac{1}{2}\exp\left(-\frac{n\overline{a^2}}{8\sigma^2}\right) \underbrace{\int\int_{\mathbb{R}^2} \frac{1}{\pi n\sigma_2\overline{a^2}/2} \exp\left(-\frac{(y_{c0} - \frac{n\overline{a^2}}{4}\sin\theta)^2 + (y_{s0} - \frac{n\overline{a^2}}{4}\cos\theta)^2}{n\sigma_2\overline{a^2}/2}\right) dy_{c0} dy_{s0}}_{=1}\right] d\theta \\
&= \frac{1}{2}\exp\left(-\frac{n\overline{a^2}}{8\sigma^2}\right)
\end{aligned}$$

where the second to last equality follows from completion of the square.

6. **Poor 3.12:**
Transmission of DPSK using a modulated sinusoidal carrier in a white Gaussian noise environment can be modelled using the same setup as Problem 3.11 under the special case that

$$\begin{aligned}
\text{send “zero”} &: \quad a_{0k} = \begin{cases} b_k & k = 1 \ldots \frac{n}{2} \\ b_{k-\frac{n}{2}} & k = \frac{n}{2}+1 \ldots n \end{cases} \\
\text{send “one”} &: \quad a_{1k} = \begin{cases} b_k & k = 1 \ldots \frac{n}{2} \\ -b_{k-\frac{n}{2}} & k = \frac{n}{2}+1 \ldots n \end{cases}
\end{aligned}$$

Here, $\{b_k, k = 1 \ldots \frac{n}{2}\}$ is the pulse shape applied to each transmission, whereas the receiver uses two subsequent pulse-intervals to make a decision. Note that $\{a_{0k}\}$ is orthogonal to $\{a_{1k}\}$, a key assumption in problem 3.11, and the energy used to transmit a single bit is $E_b = \frac{n}{2}\overline{b^2} = \frac{n}{2}\overline{a_0^2} = \frac{n}{2}\overline{a_1^2}$. From 3.11 we know that the detector has the form

$$\begin{aligned}
\text{choose “zero”} &: \quad r_1^2 \leq r_0^2 \\
\text{choose “one”} &: \quad r_1^2 > r_0^2
\end{aligned}$$

and acheives error probability

$$P_e = \frac{1}{2}\exp\left(-\frac{n\overline{a_0^2}}{8\sigma^2}\right) = \frac{1}{2}\exp\left(-\frac{E_b}{4\sigma^2}\right)$$