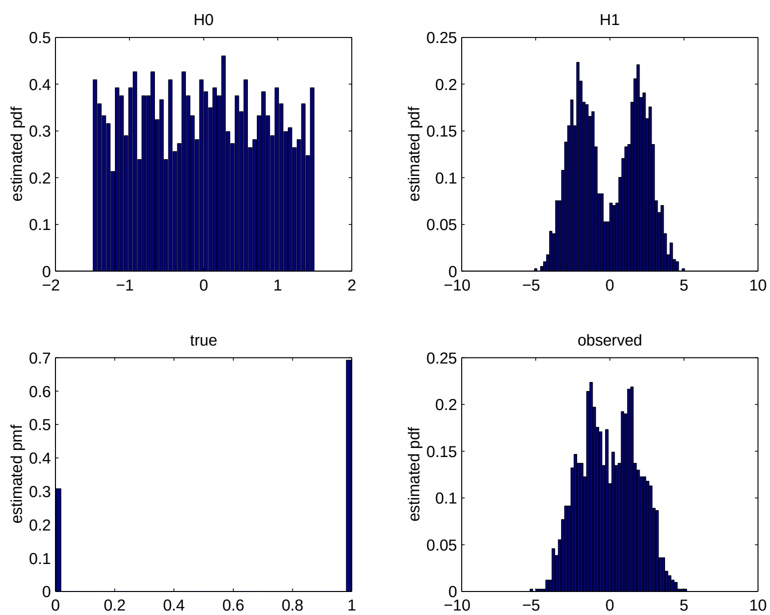
HOMEWORK SOLUTIONS #1

1. Computer Exercise 1:
   For this problem, data was generated as follows:

$$H_0 : \quad y \sim \mathrm{U}[-1.5, 1.5]$$
$$H_1 : \quad y \sim \frac{1}{2}\mathcal{N}(-2,1) + \frac{1}{2}\mathcal{N}(2,1)$$

with prior probability, $\pi_0 = 0.3$. This might have been guessed via pdf/pmf approximation using histograms:

pdf/pmf approximations:

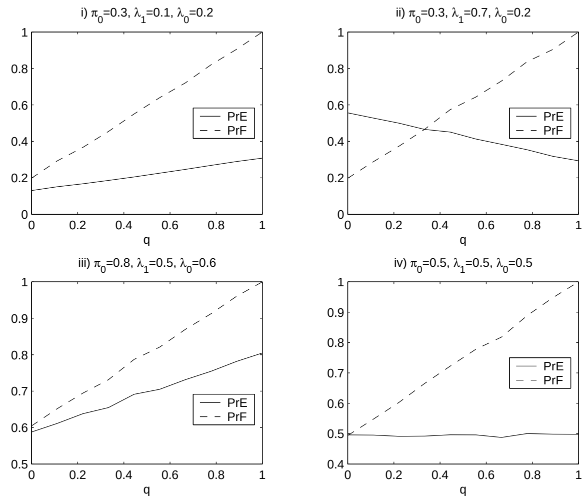


There are various ways in which one could design a decision rule. Now that you have learned more about decision theory, do you have some better ways of attacking this problem?

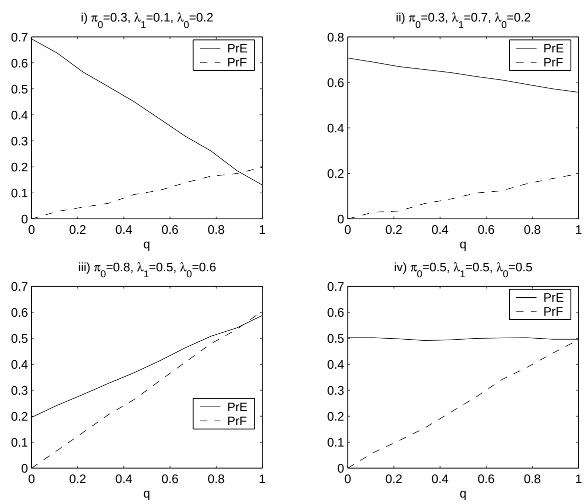
2. Computer Exercise 2:
   My program generated the following output. (Deterministic on left, random on right.)

```
>> binary
case i)
Always zero: PrE=0.692, PrF=0.000
Always one: PrE=0.308, PrF=1.000
Keep bits: PrE=0.130, PrF=0.198
Flip bits: PrE=0.870, PrF=0.802

case ii)
Always zero: PrE=0.707, PrF=0.000
Always one: PrE=0.293, PrF=1.000
Keep bits: PrE=0.556, PrF=0.198
Flip bits: PrE=0.444, PrF=0.802

case iii)
Always zero: PrE=0.195, PrF=0.000
Always one: PrE=0.805, PrF=1.000
Keep bits: PrE=0.588, PrF=0.605
Flip bits: PrE=0.412, PrF=0.395

case iv)
Always zero: PrE=0.502, PrF=0.000
Always one: PrE=0.498, PrF=1.000
Keep bits: PrE=0.496, PrF=0.494
Flip bits: PrE=0.504, PrF=0.506
```

Randomized $\delta(0)$ rule:



Randomized $\delta(1)$ rule:



3. Poor II-1:
The likelihood ratio for this binary channel problem is

$$L(y) \quad = \quad \begin{cases} \frac{\lambda_1}{1-\lambda_0} & y=0 \\ \frac{1-\lambda_1}{\lambda_0} & y=1 \end{cases}$$

Under uniform costs, the threshold is $\tau = \frac{\pi_0}{1-\pi_0}$. There are several different ways to approach this problem. I considered the possible rules and derived the Bayes risk for each rule. Thus,

$$\begin{array}{lll} \text{if } \ y=0 & \text{then } \ \delta_B(y)=1 & \text{if } \frac{\lambda_1}{1-\lambda_0} > \frac{\pi_0}{1-\pi_0} \\ & & \Leftrightarrow \text{if } \frac{\lambda_1}{1-\lambda_0+\lambda_1} > \pi_0 \\ \text{if } \ y=1 & \text{then } \ \delta_B(y)=1 & \text{if } \frac{1-\lambda_1}{\lambda_0} > \frac{\pi_0}{1-\pi_0} \\ & & \Leftrightarrow \text{if } \frac{1-\lambda_1}{1+\lambda_0-\lambda_1} > \pi_0 \end{array}$$

Using the notation in the textbook, we can define two quantities:

$$\begin{aligned} \bar{\pi} &= \max\left\{\frac{\lambda_1}{1-\lambda_0+\lambda_1}, \frac{1-\lambda_1}{1+\lambda_0-\lambda_1}\right\} \\ \underline{\pi} &= \min\left\{\frac{\lambda_1}{1-\lambda_0+\lambda_1}, \frac{1-\lambda_1}{1+\lambda_0-\lambda_1}\right\} \end{aligned}$$

From these we can determine all of the possible Bayes rules depending on the values of $\pi_0, \lambda_1, \lambda_0$:

$$\begin{array}{cl} \pi_0 < \underline{\pi} & \delta_B(y)=1 \\ \pi_0 > \bar{\pi} & \delta_B(y)=0 \\ \frac{1-\lambda_1}{1+\lambda_0-\lambda_1} \geq \pi_0 \geq \frac{\lambda_1}{1-\lambda_0+\lambda_1} & \delta_B(y)=y \\ \frac{\lambda_1}{1-\lambda_0+\lambda_1} \geq \pi_0 \geq \frac{1-\lambda_1}{1+\lambda_0-\lambda_1} & \delta_B(y)=1-y \end{array}$$

Next we determine the associated Bayes risk for each of these rules. Since we are considering the case of uniform costs, the general formula for the risk reduces to $r(\delta) = \pi_0 P_0(\Gamma_1) + (1-\pi_0)P_1(\Gamma_0)$. For each of the cases above, we'll calculate $P_i(\Gamma_j),\ i \neq j$ and then the total risk:

$$\begin{array}{clll} \pi_0 < \underline{\pi} & P_0(\Gamma_1)=1 & P_1(\Gamma_0)=0 & r(\delta)=\pi_0 \\ \pi_0 > \bar{\pi} & P_0(\Gamma_1)=0 & P_1(\Gamma_0)=1 & r(\delta)=1-\pi_0 \\ \frac{1-\lambda_1}{1+\lambda_0-\lambda_1} \geq \pi_0 \geq \frac{\lambda_1}{1-\lambda_0+\lambda_1} & P_0(\Gamma_1)=\lambda_0 & P_1(\Gamma_0)=\lambda_1 & r(\delta)=\pi_0\lambda_0+(1-\pi_0)\lambda_1 \\ \frac{\lambda_1}{1-\lambda_0+\lambda_1} \geq \pi_0 \geq \frac{1-\lambda_1}{1+\lambda_0-\lambda_1} & P_0(\Gamma_1)=1-\lambda_0 & P_1(\Gamma_0)=1-\lambda_1 & r(\delta)=\pi_0(1-\lambda_0)+(1-\pi_0)(1-\lambda_1) \end{array}$$

You can easily verify that the four risk expressions above are concisely written as:

$$r(\delta) \;=\; V(\pi_0) \;=\; \min\{(1-\pi_0)\lambda_1, \pi_0(1-\lambda_0)\} + \min\{(1-\pi_0)(1-\lambda_1), \pi_0\lambda_0\}$$

4. Poor II-2(a):
The likelihood ratio is given by

$$L(y) = \frac{p_1(y)}{p_0(y)} = \begin{cases} \frac{3}{2(y+1)} & y \in [0,1] \\ \text{undefined} & y \notin [0,1] \end{cases}.$$

With uniform costs and equal priors, the threshold $\tau = 1$. Noting that $\frac{3}{2(y+1)} \geq 1$ is equivalent to $y \leq \frac{1}{2}$, and recalling that the likelihood ratio is only defined over $y \in [0,1]$ we conclude that

$$\Gamma_1 = [0, 1/2], \quad \Gamma_0 = (1/2, 1].$$

Thus the Bayes rule is given by

$$\delta_B(y) = \begin{cases} 1 & 0 \leq y \leq 1/2 \\ 0 & 1/2 < y \leq 1 \end{cases}.$$

The corresponding minimum Bayes risk is

$$r(\delta_B) = \frac{1}{2}\int_{\Gamma_0} p_1(y)dy + \frac{1}{2}\int_{\Gamma_1} p_0(y)dy = \frac{1}{2}\int_0^{1/2} \frac{2}{3}(y+1)dy + \frac{1}{2}\int_{1/2}^1 dy = \frac{11}{24}.$$

5. Poor II-6(a):
Here we have $p_0(y) = p_N(y+s)$ and $p_1(y) = p_N(y-s)$, which gives

$$L(y) = \frac{p_1(y)}{p_0(y)} = \frac{1+(y+s)^2}{1+(y-s)^2}.$$

With equal priors and uniform costs, we find that the Bayes rule is

$$\begin{aligned} L(y) &\geq 1 \\ \Leftrightarrow 1+(y+s)^2 &\geq 1+(y-2)^2 \\ \Leftrightarrow ys &\geq 0 \\ \Leftrightarrow y &\geq 0 \qquad \text{(since } s>0) \end{aligned}$$

or, in other words,

$$\delta_B(y) = \begin{cases} 1 & \text{if } y \geq 0 \\ 0 & \text{if } y < 0 \end{cases}$$

The minimum Bayes risk is then

$$r(\delta_B) = \frac{1}{2}\int_0^{\infty} \frac{1}{\pi\left[1+(y+s)^2\right]}dy + \frac{1}{2}\int_{-\infty}^0 \frac{1}{\pi\left[1+(y-s)^2\right]}dy = \frac{1}{2} - \frac{\tan^{-1}(s)}{\pi}.$$