Non-coherent Detection of an AM Signal in Additive White Gaussian Noise:

$$H_0 \;:\; \underline{Y} = \underline{N}$$
$$H_1 \;:\; \underline{Y} = \underline{N} + \underline{s(\theta)}$$

Problem definition:

- $\underline{N} \sim \mathcal{N}(\underline{0}, \sigma^2 I)$
- $s_k(\theta) = a_k \sin((k-1)\omega_c T_s + \theta)\ k = 1, \cdots, n$
- $a_1, \cdots, a_n$ known sequence of amplitudes ("pulse shape")
- ω_c known carrier frequency
- T_s known sampling period
  (assume $n\omega_c T_s = 2\pi m$, i.e., integer # of periods in signal)
- $\Theta \sim U[0, 2\pi]$ random phase

Likelihood ratio:

$$\begin{aligned}
L(\underline{y}) &= \frac{p_1(\underline{y})}{p_0(\underline{y})} = \frac{\frac{1}{2\pi}\int_0^{2\pi} p_{1,\theta}(\underline{y})d\theta}{p_0(\underline{y})} \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{p_{1,\theta}(\underline{y})}{p_0(\underline{y})}\right]d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\exp\left\{\underline{s(\theta)}^T\Sigma^{-1}\left(\underline{y} - \frac{1}{2}\underline{s(\theta)}\right)\right\}d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\exp\left\{\frac{1}{\sigma^2}\sum_{k=1}^{n} y_k s_k(\theta) - \frac{1}{2\sigma^2}\sum_{k=1}^{n} s_k^2(\theta)\right\}d\theta
\end{aligned}$$

since $\Sigma = \sigma^2 I$.

Want to isolate θ.
Recall $\quad s_k(\theta) = a_k \sin(\,(k-1)\omega_c T_s + \theta)$.

- using $\sin(a+b) = \cos a \sin b + \sin a \cos b$
  first term:

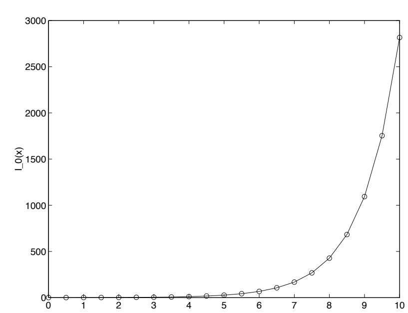
$$\begin{aligned}
\sum_{k=1}^{n} y_k s_k(\theta) &= y_c \sin\theta + y_s \cos\theta \\
y_c &:= \sum_{k=1}^{n} y_k a_k \cos((k-1)\omega_c T_s) \\
y_s &:= \sum_{k=1}^{n} y_k a_k \sin((k-1)\omega_c T_s)
\end{aligned}$$

- using $\sin^2 a = \frac{1}{2} - \frac{1}{2}\cos 2a$
  second term:

$$\begin{aligned}
-\frac{1}{2}\sum_{k=1}^{n} s_k^2(\theta) &= -\frac{1}{4}\sum_{k=1}^{n} a_k^2 + \underbrace{\frac{1}{4}\sum_{k=1}^{n} a_k^2 \cos(2(k-1)\omega_c T_s + 2\theta)}_{=0 \quad \text{(filtered double freq term)}} \\
&= -\frac{n\bar{a^2}}{4}
\end{aligned}$$

$$\text{where } \bar{a^2} := \frac{1}{n}\sum_{k=1}^{n} a_k^2$$

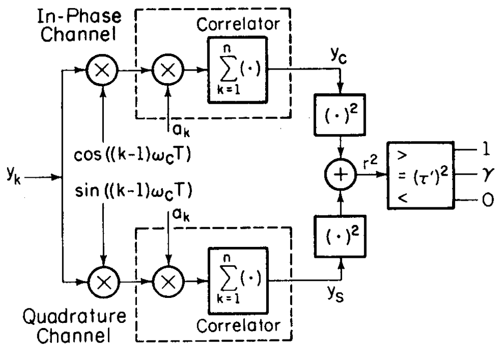
$$\begin{aligned}
\text{Thus }\; L(\underline{y}) &= \exp\left\{-\frac{n\bar{a^2}}{4\sigma^2}\right\} \\
&\quad \times \frac{1}{2\pi}\int_0^{2\pi}\exp\left\{\frac{1}{\sigma^2}(y_c\sin\theta + y_s\cos\theta)\right\}d\theta \\
&= \exp\left\{-\frac{n\bar{a^2}}{4\sigma^2}\right\} I_0\left(\frac{r}{\sigma^2}\right) \\
\text{where } r &:= \sqrt{y_c^2 + y_s^2} \\
\text{and } I_0(u) &:= \frac{1}{2\pi}\int_0^{2\pi}\exp\{u\cos(\theta - \phi)\}d\theta \quad \forall\phi \\
&= \text{modified Bessel function (see Poor p.34)}
\end{aligned}$$



Since monotonic increasing in r, the LRT takes the form:

$$r^2 = (y_c^2 + y_s^2) \;\underset{<}{\overset{>}{}}\; (\tau')^2$$

Envelope Detector:



Detector Performance:

Need to examine the statistics of

$$\left(Y_c^2 + Y_s^2\right)$$

Recalling definitions of y_c and y_s:

$$Y_c := \sum_{k=1}^{n} Y_k a_k \cos((k-1)\omega_c T_s)$$
$$Y_s := \sum_{k=1}^{n} Y_k a_k \sin((k-1)\omega_c T_s)$$

Y_k Gaussian $\Rightarrow$ Y_c and Y_s Gaussian, but statistics depend on whether H_0 or H_1 is true.

For now, condition on H_0 (where $Y_k = N_k$).

$$\mathbf{E}\{Y_c|H_0\} = \sum_{k=1}^{n} \mathbf{E}\{Y_k|H_0\}\, a_k \cos((k-1)\omega_c T_s) = 0$$
$$\mathbf{E}\{Y_s|H_0\} = \sum_{k=1}^{n} \underbrace{\mathbf{E}\{Y_k|H_0\}}_{=0}\, a_k \sin((k-1)\omega_c T_s) = 0$$

Now the variance under H_0:

$$\begin{aligned}\mathbf{Var}\{Y_c|H_0\} &= \mathbf{E}\{Y_c^2|H_0\} \\ &= \sum_{k=1}^{n}\sum_{l=1}^{n} a_k a_l \mathbf{E}\{N_k N_l\} \times \\ &\quad \cos((k-1)\omega_c T_s)\cos((l-1)\omega_c T_s) \\ &= \sigma^2 \sum_{k=1}^{n} a_k^2 \cos^2((k-1)\omega_c T_s) \\ &= \frac{\sigma^2}{2}\sum_{k=1}^{n} a_k^2 + \underbrace{\frac{\sigma^2}{2}\sum_{k=1}^{n} a_k^2 \cos(2(k-1)\omega_c T_s)}_{=0 \text{ (filtered double freq term)}} \\ &= \frac{\sigma^2 n \bar{a^2}}{2}\end{aligned}$$

and using similar arguments

$$\mathbf{Var}\{Y_s|H_0\} = \frac{\sigma^2 n \bar{a^2}}{2}$$

So, both Y_c and Y_s distributed as

$$Y_c, Y_s \mid H_0 \sim \mathcal{N}\left(0, \frac{\sigma^2 n \bar{a^2}}{2}\right)$$

What else is needed to characterize the Gaussian vector $[Y_c, Y_s]^T$?

Covariance:

$$\begin{aligned}\mathbf{Cov}\{Y_c, Y_s|H_0\} &= \mathbf{E}\{Y_c Y_s|H_0\} \\ &= \sum_{k=1}^{n}\sum_{l=1}^{n} a_k a_l \mathbf{E}\{N_k N_l\} \times \\ &\quad \cos((k-1)\omega_c T_s)\sin((l-1)\omega_c T_s) \\ &= \sigma^2 \sum_{k=1}^{n} a_k^2 \cos((k-1)\omega_c T_s)\sin((k-1)\omega_c T_s) \\ &= \frac{\sigma^2}{2}\sum_{k=1}^{n} a_k^2 \sin(2(k-1)\omega_c T_s) \\ &= 0.\end{aligned}$$

using $\sin(2b) = 2\sin(b)\cos(b)$, and assuming that double frequency term is filtered by pulse shape $\{a_k^2\}$.

Probability of False Alarm:

Know that, under H_0, have Y_c, Y_s i.i.d. $\sim \mathcal{N}(0, n\sigma^2\bar{a}^2/2)$.

Since LRT compares $y_c^2 + y_s^2$ to $(\tau')^2$, no randomization needed.

$$P_0[\Gamma_1] \;=\; \int\int_{\{y_c^2+y_s^2 \geq (\tau')^2\}} \frac{1}{\pi n\sigma^2\bar{a}^2} \exp\left\{-\frac{y_c^2+y_s^2}{n\sigma^2\bar{a}^2}\right\} dy_c\, dy_s$$

converting to polar coordinates:

$$r = \sqrt{y_c^2+y_s^2} \qquad \psi = \arctan\left(\frac{y_s}{y_c}\right)$$

plug in

$$\begin{aligned} P_0[\Gamma_1] &= \frac{1}{\pi n\sigma^2\bar{a}^2}\int_0^{2\pi}\int_{\tau'}^{\infty} r\exp\left\{-\frac{r^2}{n\sigma^2\bar{a}^2}\right\} dr\,d\psi \\ &= \frac{1}{\pi n\sigma^2\bar{a}^2}\left(\int_0^{2\pi} d\psi\right)\left(\int_{\tau'}^{\infty} r\exp\left\{-\frac{r^2}{n\sigma^2\bar{a}^2}\right\} dr\right) \\ &= \exp\left\{-\frac{(\tau')^2}{n\sigma^2\bar{a}^2}\right\} \end{aligned}$$

Probability of Detection:

Now need statistics of Y_s and Y_c under H_1.
Strategy: Condition on $\Theta = \theta$, then integrate out later.

$$\mathbf{E}\{Y_k | H_1, \Theta = \theta\} \;=\; a_k \sin((k-1)\omega_c T_s + \theta)$$

Use this, trig identities, and cancellation of double freq terms, get

$$\left[\begin{array}{c} Y_c \\ Y_s \end{array}\right]\Bigg| H_1, \theta \;\sim\; \mathcal{N}\left(\left[\begin{array}{c} \frac{n\bar{a}^2}{2}\sin\theta \\ \frac{n\bar{a}^2}{2}\cos\theta \end{array}\right], \left[\begin{array}{cc} \frac{n\bar{a}^2\sigma^2}{2} & 0 \\ 0 & \frac{n\bar{a}^2\sigma^2}{2} \end{array}\right]\right)$$

$$\begin{aligned} &p_{Y_c,Y_s}(y_c, y_s | H_1) \\ &= \int p_{Y_c,Y_s}(y_c, y_s | H_1, \theta)\, w(\theta)\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} p_{Y_c,Y_s}(y_c, y_s | H_1, \theta) d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{\pi n\bar{a}^2\sigma^2} e^{-\frac{1}{n\sigma^2\bar{a}^2}\left[\left(y_c - \frac{n\bar{a}^2}{2}\sin\theta\right)^2 + \left(y_s - \frac{n\bar{a}^2}{2}\cos\theta\right)^2\right]} d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{\pi n\bar{a}^2\sigma^2} e^{-\frac{1}{n\sigma^2\bar{a}^2}\left[y_c^2 + y_s^2 - n\bar{a}^2 y_c\sin\theta - n\bar{a}^2 y_s\cos\theta + \left(\frac{n\bar{a}^2}{2}\right)^2\right]} d\theta \\ &= p_{Y_c,Y_s}(y_c, y_s | H_0)\cdot\frac{1}{2\pi}\int_0^{2\pi} e^{\frac{1}{\sigma^2}\left[y_c\sin\theta + y_s\cos\theta - \frac{n\bar{a}^2}{4}\right]} d\theta \\ &= p_{Y_c,Y_s}(y_c, y_s | H_0)\; e^{-\frac{n\bar{a}^2}{4\sigma^2}}\; I_0\left(\frac{\sqrt{y_c^2+y_s^2}}{\sigma^2}\right) \end{aligned}$$

using $I_0\left(\frac{\sqrt{y_c^2+y_s^2}}{\sigma^2}\right) = \frac{1}{2\pi}\int_0^{2\pi} e^{\frac{1}{\sigma^2}[y_c\sin\theta + y_s\cos\theta]} d\theta$

Thus

$$\begin{aligned} P_D &= \int\int_{y_c^2+y_s^2 \geq (\tau')^2} p_{Y_c,Y_s}(y_c, y_s | H_1) dy_c dy_s \\ &= \int\int_{y_c^2+y_s^2 \geq (\tau')^2} p_{Y_c,Y_s}(y_c, y_s | H_0) e^{-\frac{n\bar{a}^2}{4\sigma^2}} I_0\left(\frac{\sqrt{y_c^2+y_s^2}}{\sigma^2}\right) dy_c dy_s \end{aligned}$$

Tranforming to polar coordinates we get

$$\begin{aligned} P_D &= \int_{\tau_0}^{\infty} x\exp\left\{-\frac{x^2+b^2}{2}\right\} I_0(bx) dx \\ &= Q(b, \tau_0) \;=\; \text{Marcum's Q function} \end{aligned}$$

with

$$\begin{aligned} b^2 &= \frac{n\bar{a}^2}{2\sigma^2} \quad \text{SNR parameter} \\ \tau_0 &= \frac{\tau'}{\sigma^2 b} \quad \text{normalized threshold} \\ x &= \frac{r}{\sigma^2 b} \quad \text{change of variable} \end{aligned}$$

For the ROC,

$$\alpha = P_F = \exp\left\{-\frac{\tau_0^2}{2}\right\} \;\Rightarrow\; \tau_0 = \left[2\log\left(\frac{1}{\alpha}\right)\right]^{\frac{1}{2}}$$

$$\Rightarrow\; P_D = Q\left(b, \left[2\log\left(\frac{1}{\alpha}\right)\right]^{\frac{1}{2}}\right)$$

Non-coherent vs. Coherent:

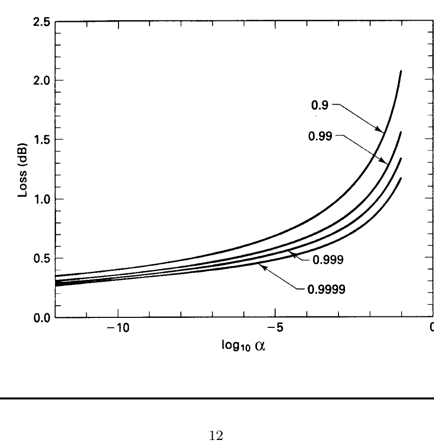
In coherent case, the key performance quantity was

$$\begin{aligned} d^2 &= \underline{s}^T\Sigma^{-1}\underline{s} \\ &= \frac{1}{\sigma^2}\sum_{k=1}^{n} a_k^2 \sin^2((k-1)\omega_c T_s + \theta) \\ &= \frac{n\bar{a}^2}{2\sigma^2} \;=\; b^2 \quad \text{(from previous slide)} \end{aligned}$$

but the P_D expression was

$$P_D \;=\; 1 - \Phi\left(\Phi^{-1}(1-\alpha) - d\right)$$

What are the values for b and d for equivalent performance?
$\rightsquigarrow$ for typical α and SNR, need $b \approx d + 0.4$.
$\rightsquigarrow$ noncoherent needs a slightly higher SNR for equal performance.



What if $\underline{s}_0 \neq \underline{0}$?

$$\begin{aligned} L(\underline{y}) &= \frac{\mathbf{E_1}\{p_N(\underline{y}-\underline{s}_1(\Theta)\}}{\mathbf{E_0}\{p_N(\underline{y}-\underline{s}_0(\Theta)\}} \\ &= \frac{\mathbf{E_1}\{p_N(\underline{y}-\underline{s}_1(\Theta)/p_N(\underline{y})\}}{\mathbf{E_0}\{p_N(\underline{y}-\underline{s}_0(\Theta)/p_N(\underline{y})\}} \\ &= \frac{L_1(\underline{y})}{L_0(\underline{y})} \\ &= \frac{e^{-\frac{n\bar{a_1^2}}{4\sigma^2}} I_0\left(\frac{r_1}{\sigma^2}\right)}{e^{-\frac{n\bar{a_0^2}}{4\sigma^2}} I_0\left(\frac{r_0}{\sigma^2}\right)} \end{aligned}$$

use $p_N(\underline{y})$ as a "catalyst"

Bayesian detection in "balanced" case:

⋆ equal priors

⋆ uniform costs

⋆ $\bar{a_1^2} = \bar{a_0^2}$

$$\text{Bayes: } \delta(\underline{y}) = \begin{cases} 1 & r_1 \geq r_0 \\ 0 & < \end{cases}$$

further, if $\underline{a}_1^t \underline{a}_0 = 0$,

$$r(\delta) = P_e = \frac{1}{2} e^{-\frac{b^2}{4}}$$