Signal Detection in Noise

In this section, we are interested in the various *structures* that arise from different assumptions on a single problem. We have vector observations that we will manipulate.

$$\begin{aligned} H_0 : \underline{Y} &= \underline{N} + \underline{S}_0 \\ H_1 : \underline{Y} &= \underline{N} + \underline{S}_1 \end{aligned}$$

in the general case where $p_N(\underline{n})$ is the noise density, we form

$$\begin{aligned} L(\underline{y}) &= \frac{p_1(\underline{y})}{p_0(\underline{y})} \\ \text{where} \quad p_i(\underline{y}) &= \mathbf{E}_{\mathbf{S}}\left\{ p_N(\underline{y} - \underline{s}) \right\} \end{aligned}$$

Consider the following cases:

1. Detection of Known/Deterministic Signals in IID Noise

$$\begin{aligned} H_0 : \underline{Y} &= \underline{N} \\ H_1 : \underline{Y} &= \underline{N} + \underline{s} \end{aligned}$$

We form the likelihood ratio,

$$\begin{aligned} L(\underline{y}) &= \prod_{k=1}^{n} L_k(y_k) \\ L_k(y_k) &= \frac{p_{N_k}(y_k - s_{1,k})}{p_{N_k}(y_k - s_{0,k})} \end{aligned}$$

Our rule is then,

$$\delta(\underline{y}) = \begin{cases} 1 & \\ \gamma & \sum_{k=1}^{n} \log L_k(y_k) \quad \begin{matrix} > \\ = \\ < \end{matrix} \quad \log \tau \\ 0 & \end{cases}$$

with most IID noise, detector has a **correlator-type** structure.

2. Locally Optimum Detection of Coherent Signals in IID Noise

$$\begin{aligned} H_0 : \underline{Y} &= \underline{N} \\ H_1 : \underline{Y} &= \underline{N} + \theta \underline{s} \\ \theta &> 0 \quad \text{unknown constant} \end{aligned}$$

$$\begin{aligned} L_\theta(\underline{y}) &= \prod_{k=1}^{n} \frac{p_{N_k}(y_k - \theta s_k)}{p_{N_k}(y_k)} \\ \left.\frac{\partial}{\partial \theta} L_\theta(\underline{y})\right|_{\theta=0} &= \sum_{k=1}^{n} s_k g_{lo}(y_k) \\ \text{where} \quad g_{lo}(x) &= -\frac{\frac{d}{dx} p_{N_k}(x)}{p_{N_k}(x)} \quad \text{locally optimal non-linearity} \end{aligned}$$

The result is a **non-linear correlator**.

3. <u>Detection of Deterministic Signals in IID Gaussian Noise</u>

   Recall that if $\underline{N} \sim \mathcal{N}\left(\underline{\mu}, \Sigma\right)$

$$p_N(\underline{n}) = \frac{1}{(2\pi)^{\frac{n}{2}}|\Sigma|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(\underline{n}-\underline{\mu})^T\Sigma^{-1}(\underline{n}-\underline{\mu})\right\}$$

   Assuming that $\Sigma > 0$, we compute the likelihood ratio, take the logarithm and absorb constants into the threshold, the resulting rule is

$$\delta(\underline{y}) = \left\{\begin{array}{ll} 1 & \\ 0 & \end{array}\right. \text{if } T(\underline{y}) = \sum_{k=1}^{n} \tilde{s}_k y_k \begin{array}{c} \geq \\ < \end{array} \tau'$$

$$\begin{aligned} \text{where } \underline{\tilde{s}} &= \Sigma^{-1}(\underline{s}_1 - \underline{s}_0) \\ \text{given } H_i \quad T(\underline{y}) &\sim \mathcal{N}\left(\underline{\tilde{s}}^T\underline{s}_i, (\underline{s}_1-\underline{s}_0)^T\Sigma^{-1}(\underline{s}_1-\underline{s}_0)\right) \end{aligned}$$

   Analysis reduces to that for scalar location testing with Gaussian error. Performance goes with $d^2 = (\underline{s}_1 - \underline{s}_0)^T\Sigma^{-1}(\underline{s}_1 - \underline{s}_0)$.

4. <u>Reduction of Dependent Gaussian Noise Case to IID Gaussian Noise Case</u>

   Recall the spectral decomposition of a matrix

$$\begin{aligned} \Sigma &= \sum_{k=1}^{n} \lambda_k \underline{v}_k \underline{v}_k^T = V\Lambda V^T \\ \lambda_k &= \text{eigenvalue} \quad \text{with} \quad \Lambda = \text{diag}[\lambda_1, \cdots, \lambda_n] \\ \underline{v}_k &= \text{eigenvector} \quad \text{with} \quad V = [\underline{v}_1, \cdots \underline{v}_n] \end{aligned}$$

   consider two decompositions of Σ

$$\begin{aligned} \Sigma &= B^2 \text{ where } B = V\Lambda^{\frac{1}{2}}V^T \\ \Sigma &= CC^T \text{ where } C = \text{Cholesky decomposition} \\ \text{form } \underline{\hat{Y}} &= B^{-1}\underline{Y} \quad \text{note} \quad B^{-1}\underline{N} \sim \mathcal{N}(\underline{0}, I) \\ \text{form } \underline{\bar{Y}} &= C^{-1}\underline{Y} \quad \text{note} \quad C^{-1}\underline{N} \sim \mathcal{N}(\underline{0}, I) \end{aligned}$$

   Cholesky decomposition offers a **causal whitening filter**.

   Note that these decompositions do not offer us anything by way of **performance.** They do offer implementation strategies.

5. <u>Signal Selection</u>

   What are the optimal choices for $\underline{s}_i$?

   Optimization criteria

$$\begin{aligned} \max d^2 &= (\underline{s}_1 - \underline{s}_0)^T\Sigma^{-1}(\underline{s}_1 - \underline{s}_0) \\ \text{subject to } \|\underline{s}_i\|^2 &\leq P \\ \text{max'ing } d^2 \rightarrow \underline{s}_1 - \underline{s}_0 &= \alpha \underline{v}_{\min} \\ \text{where } \underline{v}_{\min} &= \text{eigenvector of smallest eigenvalue, } \lambda_{\min} \quad \text{and } \alpha = \text{constant} \\ \text{power constraint } \rightarrow & \quad \|\underline{s}_i\|^2 = P \text{ and } \underline{s}_1 = -\underline{s}_0 \end{aligned}$$

   final solution

$$\begin{aligned} \underline{s}_1 &= \sqrt{P}\underline{v}_{\min} \\ d^2 &= \frac{4P}{\lambda_{\min}} \end{aligned}$$

6. Detection of Signals with Random Parameters

This is simply a composite testing problem:

$$\begin{aligned} H_0 &: \underline{Y} = \underline{N} \\ H_1 &: \underline{Y} = \underline{N} + \underline{s(\theta)} \\ \text{where } \theta &\sim p_\Theta(\theta) \end{aligned}$$

Consider the specific example of non-coherent detection of a modulated sinusoidal carrier. The relevant points (see hand-out for a more complete description) are:

- $s_k(\theta) = a_k \sin((k-1)\omega_c T_s + \theta)\ k = 1, \cdots, n$
- $\Theta \sim U[0, 2\pi]$ random phase
- use trigonometric identities,

$$\begin{aligned} \sin(a+b) &= \cos a \sin b + \sin a \cos b \\ \sin^2 a &= \frac{1}{2} - \frac{1}{2}\cos 2a \end{aligned}$$

- define new variables,

$$\begin{aligned} y_c &\doteq \sum_{k=1}^{n} y_k a_k \cos((k-1)\omega_c T_s) \\ y_s &\doteq \sum_{k=1}^{n} y_k a_k \sin((k-1)\omega_c T_s) \\ \bar{a^2} &= \frac{1}{n}\sum_{k=1}^{n} a_k^2 \\ r &= \sqrt{y_c^2 + y_s^2} \end{aligned}$$

- likelihood ratio reduces to

$$\begin{aligned} L(\underline{y}) &= \exp\left\{-\frac{n\bar{a^2}}{4\sigma^2}\right\} I_0\left(\frac{r}{\sigma^2}\right) \\ I_0 &= \text{modified Bessel function, monotonically increasing in argument} \end{aligned}$$

- performance determined by statistics of Y_c, Y_s,

$$\begin{aligned} \begin{bmatrix} Y_c|H_i \\ Y_s|H_i \end{bmatrix} &\sim \mathcal{N}\left(\begin{bmatrix} \mu_{c,i} \\ \mu_{s,i} \end{bmatrix}, \begin{bmatrix} \frac{n\bar{a^2}\sigma^2}{2} & 0 \\ 0 & \frac{n\bar{a^2}\sigma^2}{2} \end{bmatrix}\right) \\ \mu_{c,0} = \mu_{s,0} &= 0 \\ \mu_{c,1} &= \frac{n\bar{a^2}}{2}\cos\theta \\ \mu_{s,1} &= \frac{n\bar{a^2}}{2}\sin\theta \end{aligned}$$

- test is compare r to threshold $\tau' = \sigma^2 I_0^{-1}\left(e^{\frac{n\bar{a}^2}{4\sigma^2}}\right)$.

  $P_F = \exp\left(-\frac{\tau'^2}{n\bar{a}^2\sigma^2}\right)$

Note: detector is an **envelope detector**.

7. Detection of Stochastic Signals

(a) Most general, Gaussian problem:

$$\begin{aligned} H_0 &: \underline{Y} \sim \mathcal{N}\left(\underline{\mu}_0, \Sigma_0\right) \\ H_1 &: \underline{Y} \sim \mathcal{N}\left(\underline{\mu}_1, \Sigma_1\right) \\ \rightarrow \quad L(\underline{y}) &= \underbrace{\frac{1}{2}\underline{y}^T\left[\Sigma_0^{-1} - \Sigma_1^{-1}\right]\underline{y}}_{\text{quadratic part}} + \underbrace{\left[\underline{\mu}_1^T\Sigma_1^{-1} - \underline{\mu}_0^T\Sigma_0^{-1}\right]\underline{y}}_{\text{linear part}} + C \end{aligned}$$

If

- $\Sigma_0 = \Sigma_1 = \Sigma$, quadratic part is zero and we have the Gaussian location testing problem.
- $\underline{\mu}_0 = \underline{\mu}_1$, then consider *wolog* $\underline{\mu}_0 = \underline{\mu}_1 = \underline{0}$ and detector statistic is strictly quadratic since C can be absorbed into threshold.

(b) Consider,

$$\begin{aligned} H_0 &: \underline{Y} = \underline{N} \\ H_1 &: \underline{Y} = \underline{N} + \underline{S} \\ \underline{N} \sim \mathcal{N}(0, \sigma^2 I) & \qquad \underline{S} \sim \mathcal{N}(0, \Sigma_s) \\ \text{test statistic is } T(\underline{y}) &= \underline{y}^T Q \underline{y} \\ Q &= \sigma^{-2}\Sigma_s\left(\sigma^2 I + \Sigma_s\right)^{-1} \end{aligned}$$

Note: this is an **energy detector**.

(c) Relationship between Independent and Dependent Signal Cases

$$\begin{aligned} H_0 &: \underline{Y} = \underline{N} \\ H_1 &: \underline{Y} = \underline{N} + \underline{S} \\ \underline{N} \sim \mathcal{N}(0, \sigma^2 I) & \qquad \underline{S} \sim \mathcal{N}(\underline{\mu}, \Sigma_s) \\ \text{let } \hat{S}_k &\doteq \mathbf{E}_1\left\{Y_k | Y_1 = y_1, \cdots, Y_{k-1} = y_{k-1}\right\} \\ \text{and } \hat{\sigma}^2_{S_k} &\doteq \mathbf{Var}_1\left\{S_k | Y_1 = y_1, \cdots, Y_{k-1} = y_{k-1}\right\} \\ \text{then } \hat{\sigma}^2_{S_k} + \sigma^2 &= \mathbf{Var}_1\left\{Y_k | Y_1 = y_1, \cdots, Y_{k-1} = y_{k-1}\right\} \\ \log L(\underline{y}) &= \frac{1}{2}\sum_{k=1}^{n}\frac{y_k^2}{\sigma^2} - \frac{1}{2}\sum_{k=1}^{n}\frac{(y_k - \hat{S}_k)^2}{\hat{\sigma}^2_{S_k} + \sigma^2} + \frac{1}{2}\sum_{k=1}^{n}\log\frac{\sigma^2}{\hat{\sigma}^2_{S_k} + \sigma^2} \end{aligned}$$