Hypothesis Testing

Minimax Risk and Rule

Now what if we don't know the priors (π_j)?
Consider the **worst** conditional risk for a given rule; consider the **worst case** set of priors.

1. Terminology/Formulae:
   (a) risk (vs. π_0):
   $$r(\pi_0, \delta) = \pi_0 R_0(\delta) + (1 - \pi_0) R_1(\delta) = (R_0(\delta) - R_1(\delta))\pi_0 + R_1(\delta)$$
   Note that $r(\pi_0, \delta)$ is linear in π_0.

   (b) minimum risk (vs. π_0):
   $$\delta_{\pi_0} = \arg\min_{\delta} r(\pi_0, \delta)$$
   $$V(\pi_0) = \min_{\delta} r(\pi_0, \delta) = \text{min Bayes risk for } \pi_0$$
   $V(\pi_0)$ is concave down and continuous, and $V(0) = C_{11}, V(1) = C_{00}$.

   (c) minimax risk:
   $$\min_{\delta} \max_{\pi_0} r(\pi_0, \delta) = \max_{\pi_0} \min_{\delta} r(\pi_0, \delta) = \max_{\pi_0} V(\pi_0) = \min_{\delta} \max\{R_0(\delta), R_1(\delta)\}$$
   The minimax rule is the Bayes rule for the "least favorable prior" π_L, where
   $$\pi_L = \arg\max_{\pi_0} V(\pi_0)$$
   Once again we a have a *likelihood ratio test*.

2. Method for determining the rule:
   (a) Determine Bayes LRT assuming known prior π_0.
   (b) Derive $R_0(\delta_{\pi_0})$ and $R_1(\delta_{\pi_0})$.
   (c) Solve for π_L (and/or τ_L) that yield $R_0(\delta_{\pi_L}) = R_0(\delta_{\pi_L})$, or solve for π_L (and/or τ_L) that yield $\frac{dV(\pi_0)}{d\pi_0} = 0$.

3. Randomization: Necessary only if $V(\pi_0)$ is not differentiable at π_L.
   Gives conditional risk $R_j(\tilde{\delta}_{\pi_L}) = qR_j(\delta^+_{\pi_L}) + (1 - q)R_j(\delta^-_{\pi_L})$, where
   $$q = \frac{R_0(\delta^+_{\pi_L}) - R_1(\delta^+_{\pi_L})}{R_0(\delta^+_{\pi_L}) - R_1(\delta^+_{\pi_L}) + R_1(\delta^-_{\pi_L}) - R_0(\delta^-_{\pi_L})} = \frac{V'(\pi^+_L)}{V'(\pi^+_L) - V'(\pi^-_L)} \quad \text{denoting derivatives by } (\cdot)'$$
   Notes:
   - There is no need to randomize when any of the following hold:
     - $V(\pi)$ is differentiable for all $\pi \in (0, 1)$.
     - $P_0[L(y) = \tau] = P_1[L(y) = \tau] = 0 \ \forall\tau$.
     - $L(Y)$ is a continuous random variable.
   - When problem is symmetric, often have $\pi_L = \frac{1}{2}$.
   - Note: a **complete solution** to a minimax detection problem includes the explicit statement of the decision rule and/or critical region **and** a calculation of the risk incurred by the rule.