# Hypothesis Testing

*Bayes Risk and Decision Rule*

Binary hypothesis testing problem:
$$H_0 : Y \sim P_0$$
$$H_1 : Y \sim P_1$$

Given an observation $y$, how do we determine from which hypothesis ($H_0$ or $H_1$) it originated?
Bayes decision rule minimizes the average risk.

1. Terminology/Formulae

   (a) Prior probabilities
   $$\pi_j = \textit{a priori} \text{ probability that } H_j \text{ is the true source}$$

   (b) decision rule

   A partition $\Gamma = \Gamma_0 \cup \Gamma_1$ where $y \in \Gamma_0$ implies the choice $H_0$ and $y \in \Gamma_1$ implies the choice $H_1$.
   Sometimes denoted $\delta(y) = \begin{cases} 1 & y \in \Gamma_1 \\ 0 & y \in \Gamma_0 \end{cases}$.

   (c) cost function
   $$C_{ij} = \text{cost of choosing } H_i \text{ when } H_j \text{ is true}$$

   (d) conditional risk
   $$\begin{aligned} P_j(\Gamma_i) &= \text{probability of choosing } H_i \text{ given that } H_j \text{ is true} \\ R_j(\delta) &= C_{1j}P_j(\Gamma_1) + C_{0j}P_j(\Gamma_0) = \text{cost of rule } \delta \text{ given that } H_j \text{ is true} \end{aligned}$$

   (e) Bayes (or average) risk
   $$r(\delta) = \pi_0 R_0(\delta) + \pi_1 R_1(\delta)$$

   (f) Bayes decision rule
   $$\begin{aligned} \delta(y) &= \begin{cases} 1 & L(y) \geq \tau \ \text{(i.e., choose } H_0) \\ 0 & L(y) < \tau \ \text{(i.e., choose } H_1) \end{cases} \quad \text{where} \ldots \\ L(y) &= \frac{p_1(y)}{p_0(y)} = \text{likelihood ratio} \\ \tau &= \frac{\pi_0(C_{10} - C_{00})}{\pi_1(C_{01} - C_{11})} = \text{threshold} = \left.\frac{\pi_0}{\pi_1}\right|_{\text{uniform costs}} = 1\Big|_{\text{uniform costs \& equal priors}} \end{aligned}$$

   Note that this is a *likelihood ratio test* (LRT).

2. Method for determining the rule:
   (a) Calculate the likelihood ratio $L(y)$.
   (b) Try to reduce LRT to " $y \gtrless \tau'$ " or " $|y| \gtrless \tau'$ " or a test involving some other simple function of $y$.
   (c) Determine the decision regions $\Gamma_0$, $\Gamma_1$.
   (d) Calculate the risk $r(\delta)$.

3. Caveats:
   (a) If $y$ is discrete, need to determine $L(y)$ for each value of $y$.
   (b) The critical regions may be the union of **disjoint** sets.
   (c) Dont assume that the LRT has the form $y \gtrless \tau'$.
   (d) Make sure that $\Gamma_j \subset \Gamma$.

4. Note: a **complete solution** to an optimal Bayes detection problem includes the explicit statement of the decision rule and/or critical region **and** a calculation of the risk incurred by the rule.