Another Bayesian Estimation Example

Estimation of Signal Amplitude:

Problem setup:

$$Y_k = N_k + \Theta s_k, \qquad k = 1, \ldots, n$$

where

$$\underline{N} \sim \mathcal{N}(\underline{0}, \Sigma)$$
$$\Theta \sim \mathcal{N}(\mu, v^2)$$
$$\underline{s} \quad \text{known constant}$$

thus

$$\underline{Y}|\Theta \sim \mathcal{N}(\Theta \underline{s}, \Sigma)$$
$$\sim C_0 \exp\left[-\frac{1}{2}(\underline{y} - \Theta\underline{s})^t \Sigma^{-1} (\underline{y} - \Theta\underline{s})^t\right]$$

Bayes estimators based on conditional pdf $w(\theta|\underline{y})$:

$$w(\theta|\underline{y}) = \frac{p_\theta(\underline{y}) w(\theta)}{\int_\Lambda p_\theta(\underline{y}) w(\theta) d\theta}$$
$$= K(\underline{y}) \exp\left[-\frac{\theta^2}{2}\left(d^2 + \frac{1}{v^2}\right) + \theta\left(\underline{s}^t \Sigma^{-1} \underline{y} + \frac{\mu}{v^2}\right)\right]$$

where

$$d^2 = \underline{s}^t \Sigma^{-1} \underline{s}$$
$$K(\underline{y}) \quad \text{not a function of } \theta$$

Guassian $w(\theta|\underline{y})$:

Completing the square, we get

$$w(\theta|\underline{y}) = C(\underline{y}) \exp\left(-\frac{\theta^2}{2q^2} + \frac{\theta m}{q} - \frac{m^2}{2q^2}\right)$$
$$= C(\underline{y}) \exp\left(-\frac{(\theta - m)^2}{2q^2}\right)$$

where

$$q^2 = \left(d^2 + \frac{1}{v^2}\right)^{-1}$$
$$m = \left(\underline{s}^2 \Sigma^{-1} \underline{y} + \frac{\mu}{v^2}\right)\left(d^2 + \frac{1}{v^2}\right)^{-1}$$
$$C(\underline{y}) \quad \text{not a function of } \theta$$

The MMSE estimate is

$$\hat{\theta}_{\text{MMSE}}(y) = \mathrm{E}\{\Theta|\underline{y}\} = m$$

The MMSE is

$$\text{MMSE} = \mathrm{E}\Big\{\mathrm{E}[(\hat{\theta}_{\text{MMSE}}(y) - \Theta)^2 | Y = y]\Big\}$$
$$= \mathrm{E}\Big\{\mathrm{var}[\Theta | Y = y]\Big\}$$
$$= \mathrm{E}\{q^2\} = q^2$$

Note that MMSE $\Leftrightarrow$ MAP $\Leftrightarrow$ MMAE estimator since $w(\theta|\underline{y})$ is unimodal and symmetric.

Interpretations:

$$\hat{\theta}_{\text{B}} = \frac{\underline{s}^t \Sigma^{-1} \underline{y} + \mu/v^2}{d^2 + 1/v^2}$$
$$= \underbrace{\left(1 - \frac{1}{v^2 d^2 + 1}\right) \frac{\underline{s}^t \Sigma^{-1} \underline{y}}{d^2} + \left(\frac{1}{v^2 d^2 + 1}\right) \mu}_{\text{convex combination}}$$

where

$$d^2 = \underline{s}^t \Sigma^{-1} \underline{s} = n \frac{\overline{s^2}}{\sigma^2} \text{ when noise i.i.d.} = n \cdot \text{SNR}$$
$$v^2 = \text{uncertainty in prior assumption regarding } \theta = \mu$$

Note that

high $d \Rightarrow$ data is reliable

high $v \Rightarrow$ prior assumption is unreliable

and that

$$vd \to \infty \Rightarrow \hat{\theta}_{\text{B}} \to \frac{\underline{s}^t \Sigma^{-1} \underline{y}}{d^2}$$
$$vd \to 0 \Rightarrow \hat{\theta}_{\text{B}} \to \mu$$
$$vd = 1 \Rightarrow \hat{\theta}_{\text{B}} = \frac{1}{2}\left(\frac{\underline{s}^t \Sigma^{-1} \underline{y}}{d^2} + \mu\right)$$

So effect of prior knowledge decreases as $n \to \infty$.