Bayesian Estimation Example

Estimating the parameter of an exponential pdf:

Problem setup:

$$
\begin{aligned}
y \in \Lambda &= [0, \infty) \\
\theta \in \Gamma &= [0, \infty) \\
p_\theta(y) &= \begin{cases} \theta e^{-\theta y} & y \geq 0 \\ 0 & y < 0 \end{cases} \\
&\quad \text{exponential pdf with unknown parameter } \theta \\
w(\theta) &= \begin{cases} \alpha e^{-\alpha\theta} & \theta \geq 0 \\ 0 & \theta < 0 \end{cases} \\
&\quad \text{known } \alpha > 0
\end{aligned}
$$

Bayes estimators based on conditional pdf $w(\theta|y)$:

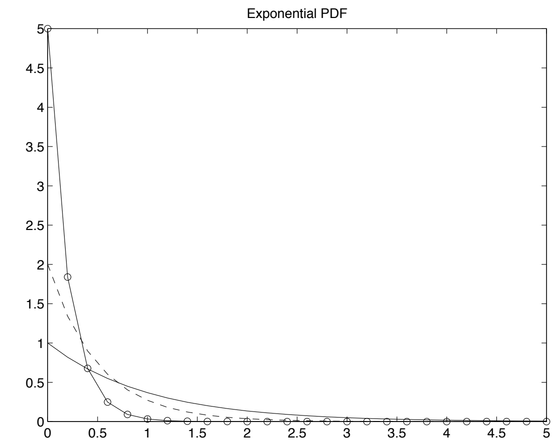
$$
\begin{aligned}
w(\theta|y) &= \frac{p_\theta(y) w(\theta)}{p(y)} \\
&= \frac{p_\theta(y) w(\theta)}{\int_\Lambda p_\theta(y) w(\theta) d\theta} \\
&= \frac{\theta e^{-\theta y} \alpha e^{-\alpha\theta}}{\int_0^\infty \theta e^{-\theta y} \alpha e^{-\alpha\theta} d\theta} \\
&= (\alpha + y)^2 \theta e^{-\theta(\alpha + y)} \quad \theta, y > 0
\end{aligned}
$$

since

$$
p(y) = \alpha \int_0^\infty \theta e^{-(y+\alpha)\theta} d\theta = \frac{\alpha}{(\alpha + y)^2}
$$

Consider the Exponential PDF

$$
p_a(x) = a e^{-ax}
$$



$\mathbf{E}\{X\} = 1/a$

$\mathbf{Var}\{X\} = 1/a^2$

$\mathbf{E}\{X^2\} = (1/a)^2 + 1/a^2 = 2/a^2$

MMSE Estimate

The **conditional mean** estimator:

$$
\begin{aligned}
\hat{\theta}_{\text{MMSE}}(y) &= \mathbf{E}\{\Theta|y\} \\
&= \int_0^\infty \theta \; w(\theta|y) \, d\theta \\
&= \int_0^\infty (\alpha + y)^2 \theta^2 e^{-\theta(\alpha+y)} d\theta \\
&= (\alpha + y) \underbrace{\int_0^\infty \theta^2 (\alpha + y) e^{-(\alpha+y)\theta} d\theta}_{= \frac{2}{(\alpha+y)^2} \text{ via } \mathrm{E}\{X^2\} \text{ property}} \\
&= \frac{2}{\alpha + y} = \frac{1}{\frac{\alpha+y}{2}} = \frac{1}{\text{avg of } y \text{ and } \alpha}
\end{aligned}
$$

What is the Bayes cost?

$$
\begin{aligned}
r(\hat{\theta}_{\text{MMSE}}) = \text{MMSE} &= \mathbf{E}\left\{\left(\hat{\theta}_{\text{MMSE}}(Y) - \Theta\right)^2\right\} \\
&= \mathbf{E}\left[\mathbf{E}\left\{\left(\hat{\theta}_{\text{MMSE}}(Y) - \Theta\right)^2 |Y\right\}\right] \\
&= \mathbf{E}\left[\mathbf{E}\left\{(\mathbf{E}\{\Theta|Y\} - \Theta)^2 |Y\right\}\right] \\
&= \mathbf{E}\left[\mathbf{Var}\{\Theta|Y\}\right]
\end{aligned}
$$

MMSE Estimate (Cont.)

Using

$$
\mathbf{Var}\{\Theta|Y\} = \mathbf{E}\{\Theta^2|Y\} - \mathbf{E}^2\{\Theta|Y\}
$$

where

$$
\begin{aligned}
\mathbf{E}\{\Theta|Y\} &= \frac{2}{\alpha + Y} \quad \text{from earlier} \\
\mathbf{E}\{\Theta^2|Y\} &= (\alpha + Y) \int_0^\infty \theta^3 (\alpha + Y) e^{-\theta(\alpha+Y)} d\theta \\
&\quad \text{and using } \int_a^b u dv = uv|_a^b - \int_a^b v du \text{ with} \\
&\quad u = \theta^3 \quad dv = (\alpha + Y) e^{-\theta(\alpha+Y)} d\theta \\
&\quad du = 3\theta^2 \quad v = -e^{-\theta(\alpha+Y)} \\
&= (\alpha + Y)\left[-\theta^3 \, e^{-\theta(\alpha+Y)}\Big|_0^\infty + \int_0^\infty 3\theta^2 e^{-\theta(\alpha+Y)} d\theta\right] \\
&= 3 \int_0^\infty \theta^2 (\alpha + Y) e^{-\theta(\alpha+Y)} d\theta \\
&= 3 \cdot \frac{2}{(\alpha + Y)^2} \quad \text{via } \mathrm{E}\{X^2\} \text{ property}
\end{aligned}
$$

we find

$$
\begin{aligned}
\mathbf{Var}\{\Theta|Y\} &= \frac{6}{(\alpha + Y)^2} - \left(\frac{2}{\alpha + Y}\right)^2 = \frac{2}{(\alpha + Y)^2} \\
\text{MMSE} &= \mathbf{E}\left[\mathbf{Var}\{\Theta|Y\}\right] \\
&= \mathbf{E}\left[\frac{2}{(\alpha + Y)^2}\right] \\
&= \int_0^\infty \frac{2}{(\alpha + y)^2} \frac{\alpha}{(\alpha + y)^2} dy = \frac{2}{3\alpha^2}
\end{aligned}
$$

MMAE Estimate

The **conditional median** estimator:

$$\hat{\theta}_{\mathsf{MMAE}}(y) = \{\hat{\theta} : \Pr[\Theta < \hat{\theta}|y] = \Pr[\Theta > \hat{\theta}|y]\}$$

Since $w(\theta|y)$ continuous,

$$\exists \hat{\theta} \text{ s.t. } \frac{1}{2} = \Pr[\Theta < \hat{\theta}|y] = \Pr[\Theta > \hat{\theta}|y]\}.$$

Thus can find MMAE estimator via

$$\begin{aligned}
\frac{1}{2} &= \int_{\hat{\theta}_{\mathsf{MMAE}}(y)}^{\infty} w(\theta|y) d\theta \\
&= \int_{\hat{\theta}_{\mathsf{MMAE}}(y)}^{\infty} (\alpha + y)^2 \theta e^{-\theta(\alpha+y)} d\theta \\
&= \left[1 + (\alpha + y)\hat{\theta}_{\mathsf{MMAE}}(y)\right] e^{-(\alpha+y)\hat{\theta}_{\mathsf{MMAE}}(y)} \\
\Rightarrow \hat{\theta}_{\mathsf{MMAE}}(y) &= \frac{T_0}{\alpha + y} \\
\text{where } T_0 \text{ solves } \frac{1}{2} &= (1 + T_0)e^{-T_0} \\
T_0 &\approx 1.68
\end{aligned}$$

Compare $\hat{\theta}_{\mathsf{MMSE}}(y)$ to $\hat{\theta}_{\mathsf{MMAE}}(y)$ ...

MAP Estimate

The **conditional mode** estimator:

$$\begin{aligned}
\hat{\theta}_{\mathsf{MAP}}(y) &= \arg\max_{\theta} \{w(\theta|y)\} \\
&= \arg\max_{\theta} \left\{\frac{p_\theta(y)w(\theta)}{p(y)}\right\} \\
&= \arg\max_{\theta} \{p_\theta(y)w(\theta)\} \\
&= \arg\max_{\theta} \{\log p_\theta(y) + \log w(\theta)\}
\end{aligned}$$

and for this example

$$\hat{\theta}_{\mathsf{MAP}}(y) = \arg\max_{\theta} \{\log\theta - \theta y + \log\alpha - \alpha\theta\}$$

Since function is concave, have global maximum.

$$\begin{aligned}
\frac{\partial}{\partial\theta}\{\log\theta - \theta y + \log\alpha - \alpha\theta\} &= \frac{1}{\theta} - (\alpha + y) = 0 \\
\Rightarrow \hat{\theta}_{\mathsf{MAP}}(y) &= \frac{1}{\alpha + y}
\end{aligned}$$

Summary

$$\begin{aligned}
\hat{\theta}_{\mathsf{MMSE}}(y) &= \frac{2}{\alpha + y} \\
\hat{\theta}_{\mathsf{MMAE}}(y) &= \frac{1.68}{\alpha + y} \\
\hat{\theta}_{\mathsf{MAP}}(y) &= \frac{1}{\alpha + y}
\end{aligned}$$

Note: all estimators of the form $\dfrac{K}{\alpha + y}$