Quiz 1 - Solutions

TOTAL - 38 points
AVERAGE - 35.8 points
MEDIAN - 36 points

1. (5 points total)

   We use Bayes formula for probability density functions:

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x)f_X(x)}{f_Y(y)}$$

   Using Stark and Woods notation:

$$\begin{aligned} f_X(x) &= \frac{1}{2}\text{rect}\left(\frac{x}{2}\right) \\ f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx \\ &= \int_{-\infty}^{\infty} f_{Y|X}(y|x)f_X(x)dx \\ &= \int_{-1}^{1} \frac{1}{2\sigma\sqrt{2\pi}} \exp\left(-\frac{(y-x)^2}{2\sigma^2}\right) dx \\ \text{let } \beta = \frac{x-y}{\sigma} & \\ &= \frac{1}{2}\int_{\frac{-1-y}{\sigma}}^{\frac{1-y}{\sigma}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\beta^2}{2}\right) d\beta \\ &= \frac{1}{2}\left[\text{erf}\left(\frac{1-y}{\sigma}\right) - \text{erf}\left(\frac{-1-y}{\sigma}\right)\right] \end{aligned}$$

   combining with the prior information:

$$f_{X|Y}(x|y) = \frac{\frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(y-x)^2}{2\sigma^2}\right) \text{rect}\left(\frac{x}{2}\right)}{\text{erf}\left(\frac{1-y}{\sigma}\right) - \text{erf}\left(\frac{-1-y}{\sigma}\right)}$$

2. (5 points total)

   (a) To solve for c;

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f_X(x)dx \\ &= c\int_0^{\infty} e^{-2x}dx = \frac{c}{2}\int_0^{\infty} e^{-u}du = \frac{c}{2} \\ \rightarrow \quad c &= 2 \end{aligned}$$

(b)

$$P[X \geq x+a] \;=\; 2\int_{x+a}^{\infty} e^{-2u}du = e^{-2(x+a)}$$

for $x > 0, a > 0$.

(c)

$$\begin{aligned} P[X \geq x+a|X>a] &= \frac{P[X \geq x+a, X>a]}{P[X>a]} \\ \text{but}\;\; P[X \geq x+a, X>a] &= P[X \geq x+a] \\ \rightarrow\;\; P[X \geq x+a|X>a] &= \frac{P[X \geq x+a]}{P[X>a]} = \frac{e^{-2(x+a)}}{e^{-2a}} = e^{-2x} \end{aligned}$$

Thus this conditional probability is **independent** of a. This why the exponential density is considered to be *memoryless.*

3. (5 points)

The cumulative distribution function for a standard Gaussian random variable ($\mathcal{N}(0,1)$) is given by

$$\Phi(x) \;=\; P[X<x] = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp -\frac{y^2}{2} dy$$

The error function is defined as,

$$\text{erf}(x) \;=\; \int_{0}^{x} \frac{2}{\sqrt{\pi}} \exp -y^2 dy$$

for $x \geq 0$

$$\begin{aligned} \Phi(x) &= \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} \exp -\frac{y^2}{2} dy + \int_{0}^{x} \frac{1}{\sqrt{2\pi}} \exp -\frac{y^2}{2} dy \\ &= \frac{1}{2} + \int_{0}^{\frac{x}{\sqrt{2}}} \sqrt{2}\frac{1}{\sqrt{2\pi}} \exp -z^2 dz \\ &= \frac{1}{2} + \frac{1}{2}\int_{0}^{\frac{x}{\sqrt{2}}} \frac{2}{\sqrt{\pi}} \exp -z^2 dz \\ &= \frac{1}{2}\left[1 + \text{erf}\left(\frac{x}{\sqrt{2}}\right)\right] \end{aligned}$$

for $x < 0$

$$\begin{aligned} \Phi(x) &= \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} \exp -\frac{y^2}{2} dy - \int_{x}^{0} \frac{1}{\sqrt{2\pi}} \exp -\frac{y^2}{2} dy \\ &= \frac{1}{2} - \int_{\frac{x}{\sqrt{2}}}^{0} \sqrt{2}\frac{1}{\sqrt{2\pi}} \exp -z^2 dz \\ &= \frac{1}{2} - \frac{1}{2}(-1)\int_{0}^{\frac{x}{\sqrt{2}}} \frac{2}{\sqrt{\pi}} \exp -z^2 dz \\ &= \frac{1}{2}\left[1 + \text{erf}\left(\frac{x}{\sqrt{2}}\right)\right] \end{aligned}$$

4. (8 points total)

   (a) Let $\underline{X}$ be a random vector (of dimension $N \times 1$) with the Gaussian density, $\mathcal{N}(\underline{m}, C)$. What is the mean vector and covariance matrix of $A\underline{X} + \underline{b}$, where A is a constant matrix of dimension $N \times N$ and $\underline{b}$ is a constant vector of dimensions $N \times 1$..

$$\begin{aligned} \mathbf{E}\{A\underline{X} + \underline{b}\} &= A\underline{m} + \underline{b} \\ \mathbf{Cov}\{A\underline{X}\} &= ACA^T \end{aligned}$$

   (b) Let $p_i(\underline{Y})$ be the Gaussian multivariate density $\mathcal{N}(\underline{s}_i, \Sigma)$ where $i = 1, 2$. Determine the ratio of the two densities : $\frac{p_1(\underline{Y})}{p_2(\underline{Y})}$. Simplify the expression as much as possible. Assume that $\underline{s}_1 \neq \underline{s}_2$.

$$\begin{aligned} \frac{p_1(\underline{Y})}{p_2(\underline{Y})} &= \frac{\frac{1}{(2\pi)^{\frac{N}{2}}|\Sigma|^{\frac{1}{2}}} \exp -\frac{1}{2}(\underline{y} - \underline{s}_1)^T \Sigma^{-1}(\underline{y} - \underline{s}_1)}{\frac{1}{(2\pi)^{\frac{N}{2}}|\Sigma|^{\frac{1}{2}}} \exp -\frac{1}{2}(\underline{y} - \underline{s}_2)^T \Sigma^{-1}(\underline{y} - \underline{s}_2)} \\ &= \frac{\exp -\frac{1}{2}(\underline{y} - \underline{s}_1)^T \Sigma^{-1}(\underline{y} - \underline{s}_1)}{\exp -\frac{1}{2}(\underline{y} - \underline{s}_2)^T \Sigma^{-1}(\underline{y} - \underline{s}_2)} \\ &= \exp -\frac{1}{2}\left[(\underline{y} - \underline{s}_1)^T \Sigma^{-1}(\underline{y} - \underline{s}_1) - (\underline{y} - \underline{s}_2)^T \Sigma^{-1}(\underline{y} - \underline{s}_2)\right] \\ &= \exp -\frac{1}{2}\left[-2y^T \Sigma^{-1}\underline{s}_1 + \underline{s}_1^T \Sigma^{-1}\underline{s}_1 + 2y^T \Sigma^{-1}\underline{s}_2 - \underline{s}_2^T \Sigma^{-1}\underline{s}_2\right] \\ &= \exp\left[y^T \Sigma^{-1}(\underline{s}_1 - \underline{s}_2) + \frac{1}{2}(\underline{s}_2^T \Sigma^{-1}\underline{s}_2 - \underline{s}_1^T \Sigma^{-1}\underline{s}_1)\right] \\ &= \exp\left[\left(y^T - \frac{1}{2}(\underline{s}_1^T + \underline{s}_2^T)\right)\Sigma^{-1}(\underline{s}_1 - \underline{s}_2)\right] \end{aligned}$$

   I **really** wanted this as simplified as possible and so I took off a point if you didn't simplify the expression enough.

5. (7 points) Assume you had a likelihood ratio test of the form:

$$(y - 7)^2 \geq \tau \rightarrow \text{choose } H_1 \qquad -\infty \leq y \leq \infty$$

Describe the decision regions Γ_1 and Γ_0 as functions of subsets of the real line.

Note that we need to consider **all** possible values of τ:

$$\text{if } \tau < 0 \rightarrow \Gamma_0 = \emptyset \text{ and } \Gamma_1 = \mathcal{R}$$

$$\begin{aligned} \text{if } \tau \geq 0 \rightarrow \Gamma_1 &= \{y : (y - 7)^2 \geq \tau\} \\ &= \{y : -\sqrt{\tau} + 7 \geq y \text{ or } y \geq \sqrt{\tau} + 7\} \\ &= (-\infty, -\sqrt{\tau} + 7] \cup [\sqrt{\tau} + 7, \infty) \\ \text{and } \Gamma_0 &= \Gamma_1^C = (-\sqrt{\tau} + 7, \sqrt{\tau} + 7) \end{aligned}$$

Now consider

$$\ln(y-7) \geq \tau \rightarrow \text{choose } H_1 \qquad -\infty \leq qy \leq \infty$$

In this case we know that $\ln$ is a monotonic function of its arguement and that we do not need to distinguish between $\tau > 0$ and $\tau < 0$ (both values are valid). However, we do note that $\ln$ is not defined for negative arguments. This will impose a limit on the decision regions.

$$\begin{aligned} \Gamma_1 &= \{y : \ln(y-7) \geq \tau\} \\ &= \{y : y \geq e^\tau + 7\} = [e^\tau + 7, \infty) \\ \text{but } \Gamma_0 &= \{y : 7 \leq y < e^\tau + 7\} = [7, e^\tau + 7) \end{aligned}$$

You do need to concern yourself with which set the *end-points* get assigned. That is, for example, $e^\tau + 7$ cannot belong to Γ_1 and Γ_0 simultaneously.

6. (3 points) The important topics from EE 804 are:

   - manipulating probability density functions
   - conditional and joint probability density functions
   - functions of random variables
   - multi-dimensional Gaussian variables

7. (5 points)

   Two Hypothesis Testing problems.

   I was, in general, very pleased with these. Most of you wrote up very interersting background statements and devised interesting hypothesis testing problems. Neat!

   Remember to distinguish between **estimation** problems (the unknown θ is drawn from a continuous set) versus **detection** problems (the unknown θ is drawn from a discrete set).