On HW#3, Problem 4

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Some students asked why a particular application of auxiliary variables and Jacobian transformation seemed to give the wrong answer to problem 4 on HW#3. I have prepared the answer below.

Problem: Let H and S be two independent variables, where $H \sim \mathcal{N}(0, \sigma^2)$ and S is discrete with $P\{S = 1\} = P\{S = -1\} = 0.5$. Let $R = HS$. Find the pdf of R.

Attempt 1: Say that

$$\begin{aligned} R &= g_1(H,S) = HS && (1) \\ W &= g_2(H,S) = S && (2) \end{aligned}$$

where W is an auxiliary variable. Given any pair $\{r, w\}$, note that the pair $\{h, s\}$ which gives $r = g_1(h, s)$ and $w = g_2(h, s)$ is uniquely given by $\{h, s\} = \{r/w, w\}$. Since W never equals zero, the quantity r/w will always be well defined. Then we can use a Jacobian technique to claim

$$\begin{aligned} f_{R,W}(r,w) &= \left| \det \begin{pmatrix} \frac{dg_1}{dh} & \frac{dg_1}{ds} \\ \frac{dg_2}{dh} & \frac{dg_2}{ds} \end{pmatrix}_{\{h,s\}=\{r/w,w\}} \right|^{-1} f_{H,S}(r/w, w) && (3) \\ &= \left| \det \begin{pmatrix} s & h \\ 0 & 1 \end{pmatrix}_{\{h,s\}=\{r/w,w\}} \right|^{-1} f_H(r/w) f_S(w) && (4) \\ &= \frac{1}{|w|} f_H(r/w) f_S(w) && (5) \\ &= \frac{1}{|w|} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(r/w)^2}{2\sigma^2}} \left(\frac{1}{2}\delta(w-1) + \frac{1}{2}\delta(w+1) \right) && (6) \end{aligned}$$

and then integrate out the auxiliary variable to obtain the answer:

$$\begin{aligned} f_R(r) &= \int_{-\infty}^{\infty} f_{R,W}(r,w) dw && (7) \\ &= \int_{-\infty}^{\infty} \frac{1}{|w|} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(r/w)^2}{2\sigma^2}} \left(\frac{1}{2}\delta(w-1) + \frac{1}{2}\delta(w+1) \right) dw && (8) \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{r^2}{2\sigma^2}} && (9) \end{aligned}$$

This is the correct answer, and a fine approach (though different from the approach taken in the solutions).

Attempt 2: Instead say that

$$\begin{aligned} R &= g_1(H,S) = HS && (10) \\ W &= g_2(H,S) = H && (11) \end{aligned}$$

where W represents a different choice for the auxiliary variable. Given any pair $\{r, w\}$, note that the pair $\{h, s\}$ which gives $r = g_1(h, s)$ and $w = g_2(h, s)$ has solutions

$$\{h,s\} = \begin{cases} \{w, r/w\}, & \text{for } w \neq 0 \\ \{0, \mathbb{R}\}, & \text{for } w = 0. \end{cases} \quad (12)$$

In other words, any $s \in \mathbb{R}$ will give $r = 0$ when $w = h = 0$. Since our equations have an uncountable number of roots, we cannot use the Jacobian technique! (Recall the countable root requirement in the notes.) This is where the confusion was coming from.

Attempt 3: This is perhaps the simplest way to solve the problem:

$$\begin{aligned} F_R(r) &= P[R \leq r] && (13) \\ &= P[R \leq r | S = 1]P[S = 1] + P[R \leq r | S = -1]P[S = -1] && (14) \\ &= P[H \leq r]P[S = 1] + P[-H \leq r]P[S = -1] && (15) \\ &= P[H \leq r]P[S = 1] + P[H \geq -r]P[S = -1] && (16) \\ &= \frac{1}{2}\int_{-\infty}^{r} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{h^2}{2\sigma^2}}\, dh + \frac{1}{2}\int_{-r}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{h^2}{2\sigma^2}}\, dh && (17) \\ &= \int_{-\infty}^{r} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{h^2}{2\sigma^2}}\, dh && (18) \end{aligned}$$

where the last equality follows from symmetry. Then

$$\begin{aligned} f_R(r) &= \frac{d}{dr} F_R(r) && (19) \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{r^2}{2\sigma^2}} && (20) \end{aligned}$$

where the last equality follows from a trivial application of Liebnitz rule (known in this form as the Fundamental Theorem of Calculus).