b) Uncorrelated means $0 = \text{cov}(X,Y) = E[(X-E[X])(Y-E[Y])] = E[XY] - E[X]E[Y]$

Here

$$E[X] = \sum_{i=1}^{5} w_i P(w_i) = (-1)\tfrac{1}{5} + (-\tfrac{1}{2})\tfrac{1}{5} + (0)\tfrac{1}{5} + (\tfrac{1}{2})\tfrac{1}{5} + (1)\tfrac{1}{5} = 0$$

$$E[Y] = \sum_{i=1}^{5} w_i^2 P(w_i) = (1)\tfrac{1}{5} + (\tfrac{1}{4})\tfrac{1}{5} + (0)\tfrac{1}{5} + (\tfrac{1}{4})\tfrac{1}{5} + (1)\tfrac{1}{5} = \tfrac{1}{10} + \tfrac{2}{5} = \tfrac{1}{2}$$

$$E[XY] = \sum_{i=1}^{5} w_i \cdot w_i^2 P(w_i) = \sum_{i=1}^{5} w_i^3 P(w_i) = (-1)\tfrac{1}{5} + (-\tfrac{1}{8})\tfrac{1}{5} + (0)\tfrac{1}{5} + (\tfrac{1}{8})\tfrac{1}{5} + (1)\tfrac{1}{5} = 0$$

Thus $\text{cov}(X,Y) = E[XY] - E[X]E[Y] = 0 - \frac{3}{10}\cdot 0 = 0$ ✓

c) For X & Y independent, we need $f_{Y|X}(y|x) = f_Y(y)$

But note that $f_Y(y) = \frac{1}{5}\delta(y) + \frac{2}{5}\delta(y-\frac{1}{4}) + \frac{2}{5}\delta(y-1)$

whereas $f_{Y|X}(y|x) = \delta(y-x^2) \neq f_Y(y)$. So they are <u>not</u> independent.

In other words, knowing X implies we know Y.

4) $P[Y=1] = \int_1^\infty f_X(x)\,dx$, $\quad P[Y=0] = \int_{-1}^{1} f_X(x)\,dx$, $\quad P[Y=-1] = \int_{-\infty}^{-1} F_X(x)\,dx$

$= Q(1)$ $\quad = Q(-1) - Q(1)$ $\quad = Q(-\infty) - Q(-1) = 1 - Q(-1)$

Thus $f_Y(y) = Q(1)\,\delta(y-1) + (Q(-1) - Q(1))\,\delta(y) + \overbrace{(1 - Q(-1))}^{Q(1)}\,\delta(y+1)$

b) $$\int_a^\infty \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}}\,dz = \int_{a-\mu}^{\infty} \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma} e^{-y^2/2\sigma^2}\,dy \quad \text{using } y = z-\mu$$

$$= \int_{\frac{a-\mu}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx \quad \text{using } x = y/\sigma,\ dx = \frac{dy}{\sigma}$$

$$= Q\left(\frac{a-\mu}{\sigma}\right)$$

c) $$P[Y=1] = \int_1^\infty \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\left(\frac{(z-\mu)^2}{2\sigma^2}\right)}\,dz = Q\left(\frac{1-\mu}{\sigma}\right)$$

$$P[Y=0] = \int_{-1}^{1} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\left(\frac{(z-\mu)^2}{2\sigma^2}\right)}\,dz = Q\left(\frac{-1-\mu}{\sigma}\right) - Q\left(\frac{1-\mu}{\sigma}\right)$$

$$P[Y=-1] = \int_{-\infty}^{-1} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\left(\frac{(z-\mu)^2}{2\sigma^2}\right)}\,dz = 1 - Q\left(\frac{-1-\mu}{\sigma}\right)$$

Thus $f_Y(y) = Q\left(\frac{1-\mu}{\sigma}\right)\delta(y-1) + \left[Q\left(\frac{-1-\mu}{\sigma}\right) - Q\left(\frac{1-\mu}{\sigma}\right)\right]\delta(y) + \left[1 - Q\left(\frac{-1-\mu}{\sigma}\right)\right]\delta(y+1)$