E804 MIDTERM SOLUTIONS PHIL SCHNITER

(1) Probability of burning at least T months = Probability of failing after T months

$P[T>t] = P[T>t \mid A]\,P[A] + P[T>t \mid B]\,P[B]$

$= (1 - P[T \le t \mid A])\,P[A] + (1 - P[T \le t \mid B])\,P[B]$

$= \boxed{e^{-t/2} \cdot \frac{1}{4} + e^{-t/5} \cdot \frac{3}{4}}$ for $t \ge 0$

(2) $E[X] = \left(\frac{1}{j}\right)^1 \frac{\partial}{\partial \omega_1} \Phi_{XY}(\omega_1,\omega_2)\Big|_{\omega_1=\omega_2=0} = (-j)(-1)(1-j\omega_1)^{-2}(1-2j\omega_2)^{-1}(-j)\Big|_{\omega_1=\omega_2=0} = \boxed{1}$

$E[Y] = \left(\frac{1}{j}\right)^1 \frac{\partial}{\partial \omega_2} \Phi_{XY}(\omega_1,\omega_2)\Big|_{\omega_1=\omega_2=0} = (-j)(1-j\omega_1)^{-2}(-1)(1-2j\omega_2)^{-2}(-2j)\Big|_{\omega_1=\omega_2=0} = \boxed{2}$

b) $E[XY] = \left(\frac{1}{j}\right)^2 \frac{\partial}{\partial \omega_1 \partial \omega_2} \Phi_{XY}(\omega_1,\omega_2)\Big|_{\omega_1=\omega_2=0} = (-1)\frac{\partial}{\partial \omega_2}\left[(-1)(1-j\omega_1)^{-2}(1-2j\omega_2)^{-1}(-j)\right]\Big|_{\omega_1=\omega_2=0}$

$= (-1)(-1)(1-j\omega_1)^{-2}(-1)(1-2j\omega_2)(-2j)(-j)\Big|_{\omega_1=\omega_2=0} = 2$

$\rho(X,Y) = E[XY] - E[X]E[Y] = 1 \cdot 2 - 2 = \boxed{0}$

(3) Given $X(\omega_i) = \omega_i$ $(1 \le i \le 5)$ for $\Omega = \{\omega_1, \dots, \omega_5\}$, we need that every Borel set has inverse image contained in $\mathcal{F}$, whose elements are subsets of Ω. As we can construct Borel sets that contain any combination of the elements in Ω as well as elements outside of Ω, we need $\mathcal{F}$ to be the <u>power set</u> of Ω. I.e., the set of all combinations of subsets of Ω. Such an $\mathcal{F}$ will contain

$$\underbrace{\emptyset}_{\text{empty set}} + \underbrace{\binom{5}{1}}_{\text{\# sets with one element}} + \underbrace{\binom{5}{2}}_{\text{\# sets with two elements}} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5} \text{ sets,}$$

$$= 1 + \frac{5\cdot4\cdot3\cdot2\cdot1}{(1)(4\cdot3\cdot2\cdot1)} + \frac{5\cdot4\cdot3\cdot2\cdot1}{(2\cdot1)(3\cdot2\cdot1)} + \frac{5\cdot4\cdot3\cdot2\cdot1}{(3\cdot2\cdot1)(2\cdot1)} + \frac{5\cdot4\cdot3\cdot2\cdot1}{(4\cdot3\cdot2\cdot1)(1)} + \frac{5!}{5!\,1!}$$

$$= 1 + 5 + 10 + 10 + 5 + 1 = \boxed{32}$$

b) Note that $Y(\omega_i) = \omega_i^2$ has range $\{0, \frac{1}{4}, 1\}$, and inverse image will be contained in $\mathcal{F}$.