EE804 HW #6 SOLUTIONS PHIL SCHNITER

(1) $Re^{j\theta} = R\cos\theta + jR\sin\theta$. Say $X = R\cos\theta$ & $Y = R\sin\theta$

Then given $(X,Y) = (x,y)$, we have unique roots

$$r = \sqrt{x^2+y^2}, \quad \theta = \begin{cases} \tan^{-1}(y/x) + \pi/2 & x > 0 \\ \tan^{-1}(y/x) + 3\pi/2 & x < 0 \end{cases}$$

So we can use the Jacobian technique

$$J = \begin{vmatrix} \cos\theta & -R\sin\theta \\ \sin\theta & R\cos\theta \end{vmatrix} = R(\cos^2\theta + \sin^2\theta) = R$$

Thus

$$f_{X,Y}(x,y) = |J^{-1}(r,\theta)|\, f_{R,\theta}(r,\theta)\Big|_{(r,\theta) = \text{roots given } (x,y)}$$

$$= \frac{1}{\sqrt{x^2+y^2}} \cdot \frac{\sqrt{x^2+y^2}}{\sigma^2} e^{-(x^2+y^2)/2\sigma^2} u(x^2+y^2) \cdot \frac{1}{2\pi}$$

$$= \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}x^2/\sigma^2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}y^2/\sigma^2}$$

and it can be seen that X & Y are jointly Gaussian & Independent. Since the real and imaginary parts of $Re^{j\theta}$ are jointly Gaussian, independent, and of equal variance, $Re^{j\theta}$ is circular complex Gaussian.

(2) We know that $\forall \varepsilon > 0$, $\left\{ \begin{array}{l} \Pr(|X_n - X| > \varepsilon/\sqrt{2}) \to 0 \\ \Pr(|Y_n - Y| > \varepsilon/\sqrt{2}) \to 0 \end{array} \right\}$

meaning $\left[\Pr(|X_n - X| > \varepsilon/\sqrt{2}) + \Pr(|Y_n - Y| > \varepsilon/\sqrt{2})\right] \to 0$

Since

$$\Pr(|X_n - X| > \varepsilon/\sqrt{2}) + \Pr(|Y_n - Y| > \varepsilon/\sqrt{2}) \geq \Pr(|X_n - X| > \varepsilon/\sqrt{2}, |Y_n - Y| > \varepsilon/\sqrt{2})$$
$$\geq \Pr(|X_n - X + Y_n - Y| > \varepsilon)$$

Then $\forall \varepsilon > 0$, $\Pr(|X_n + Y_n - (X+Y)| > \varepsilon) \to 0$

meaning $X_n + Y_n \xrightarrow{P} X + Y$