EE804 HW #5 SOLUTIONS PHIL SCHNITER

1) $f_X(x) = \lambda e^{-\lambda x} u(x)$, $E[X] = \frac{1}{\lambda}$, $var(X) = \frac{1}{\lambda^2}$, $P[X \geq a] = 1 - P[X < a] = \begin{cases} 1 & a \leq 0 \\ e^{-\lambda a} & a > 0 \end{cases}$

i) Markov : $P[X \geq a] \leq \frac{E[X]}{a} = \boxed{\frac{1}{\lambda a}}$ for $\boxed{a \geq 0}$

ii) Chebyshev : $P[|X - \mu| \geq b] = P[X - \mu \geq b] + P[X - \mu \leq -b]$
$= P[X \geq \mu + b] + \underbrace{P[X \leq \mu - b]}_{=0 \text{ when } b \geq \mu}$

$a = \mu + b \Rightarrow P[X \geq a] = P[|X - \mu| \geq a - \mu]$ when $\boxed{a \geq 2\mu}$

$\leq \frac{var(X)}{(a-\mu)^2} = \boxed{\frac{1}{\lambda^2 (a - \lambda^{-1})^2}}$ by Chebyshev Inequality

iii) Chernoff : $\Theta_X(t) = \int_0^\infty f_X(x) e^{tx} dx = \lambda \int_0^\infty e^{(t-\lambda)x} dx = \frac{\lambda}{t-\lambda}\left[e^{(t-\lambda)\infty} - 1\right] = \frac{\lambda}{\lambda - t}$

$e^{-at}\Theta_X(t) = e^{-at}\lambda/(\lambda - t)$ for $t < \lambda$ for $t - \lambda < 0$

Differentiate to find $\min_{t>0} e^{-at}\Theta_X(t)$

$\frac{\partial}{\partial t}\left[e^{-at}\Theta_X(t)\right] = e^{-at}\left(\frac{-a\lambda(\lambda - t) + \lambda}{(\lambda - t)^2}\right) = 0$ when $t = \lambda - \frac{1}{a}$

Recall that we need $t > 0$ & $t < \lambda$. When $t = \lambda - \frac{1}{a}$, implies $\boxed{a > \frac{1}{\lambda}}$

$P[X \geq a] \leq \min_{\substack{t>0 \\ t<\lambda}} e^{-at}\Theta_X(t) = e^{-a(\lambda - \frac{1}{a})}\frac{\lambda}{\lambda - (\lambda - \frac{1}{a})} = \boxed{e^{-a\lambda + 1} a\lambda}$

See plots on following pages ...

2) $E[Y] = E[\underline{a}^t \underline{X} + b] = \underline{a}^t E[\underline{X}] + b = \underline{a}^t \underline{\mu} + b = (2, -1, 2)\begin{pmatrix} 5 \\ -5 \\ 6 \end{pmatrix} + 5 = 32$

$var(Y) = E[(Y - E[Y])^2] = E[(\underline{a}^t \underline{X} + b - \underline{a}^t \underline{\mu} - b)^2] = E[(\underline{a}^t(\underline{X} - \underline{\mu}))^2]$

$= \underline{a}^t E[(\underline{X} - \underline{\mu})(\underline{X} - \underline{\mu})^t]\underline{a} = \underline{a}^t K \underline{a} = 25$

3) a) Must have $K_{XX} = K_{XX}^t$

b) Diagonal elements of K_{XX} are variances, which must be positive

c) Off-diagonal elements of K_{XX} are $cov(X_k, X_l)$, which are $\mathbb{R}$ when $\underline{X} \in \mathbb{R}^3$

d) By the Schwarz inequality

$cov^2(X_1, X_2) = E^2((X_1 - \mu_1)(X_2 - \mu_2)) \leq E((X_1 - \mu_1)^2) E((X_2 - \mu_2)^2) = var(X_1) var(X_2)$

but in this example, $cov^2(X_1, X_2) = (-4)^2 = 16$ and $var(X_1) var(X_2) = 2 \cdot 3 = 6$

e) Again, (a) fails since now $K_{XX} = K_{XX}^H$ and (b) & (d) fail as before

But (c) satisfies $K_{XX} = K_{XX}^H$ and is positive definite, thus a valid covmtx