EE 804 HW #4 SOLUTIONS P. SCHNITER

1) $\left\{ \begin{matrix} z = x+y \\ u = x/y \end{matrix} \right\}$ has unique roots $\left\{ \begin{matrix} x = zu/(u+1) \\ y = z/(u+1) \end{matrix} \right\}$

Jacobian $J = \det \begin{bmatrix} dz/dx & dz/dy \\ du/dx & du/dy \end{bmatrix} = \det \begin{bmatrix} 1 & 1 \\ 1/y & -x/y^2 \end{bmatrix} = -\frac{x}{y^2} - \frac{1}{y} = -\frac{x+y}{y^2}$

evaluated at roots: $= -(u+1)^2/z$

$$f_{Z,U}(z,u) = \begin{cases} 2\exp\left(-\frac{z(u+2)}{u+1}\right) \left| -(u+1)^2/z \right|^{-1} = \frac{2z}{(u+1)^2} \exp\left(-\frac{z(u+2)}{u+1}\right) & z, u > 0 \\ 0 & \text{else} \end{cases}$$

2) If $\underline{X}$ is dependent, $R$ is non-diagonal.
Since $R$ is positive definite symmetric, know eigendecomposition exists giving $R = V \Lambda V^t$ for unitary $V$ and positive diagonal $\Lambda$.
Know linear transformation $\underline{Y} = W \underline{X}$ gives $\underline{Y} \sim N(\underline{0}, WRW^t)$ where we want $WRW^t = \sigma^2 I$ for $I$ the identity matrix.
Want $\sigma^2 I = WV\Lambda V^t W^t = (WV\Lambda^{1/2}\sigma^{-1})\, \sigma^2 I\, (WV\Lambda^{1/2}\sigma^{-1})^t$
where it is now clear that one choice for $W$ would be

$$W = (V\Lambda^{1/2}\sigma^{-1})^{-1} = \sigma \Lambda^{-1/2} V^t$$

b) When $R = \begin{pmatrix} 5 & -\sqrt{3} \\ -\sqrt{3} & 7 \end{pmatrix}$ we have $V = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$, $\Lambda = \begin{pmatrix} 4 & 0 \\ 0 & 8 \end{pmatrix}$

which can be found via matlab, or by hand solving $R\underline{v} = \lambda \underline{v}$ for $\{\lambda, \underline{v}\}$.
Then

$$W = 1 \cdot \begin{pmatrix} 1/2 & 0 \\ 0 & \sqrt{2}/4 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/4 & 1/4 \\ -\sqrt{2}/8 & \sqrt{6}/8 \end{pmatrix}$$

3) From HW 2, #7, know $f_X(x) = f_Y(x) = 3\sqrt{x} - 3x^2$

a) Thus $E[X] = E[Y] = \int_0^1 x(3\sqrt{x} - 3x^2)\,dx = 3\left(\frac{2}{5}x^{5/2} - \frac{1}{4}x^4\right)\Big|_0^1 = \frac{9}{20}$

b) $E[X^2] = E[Y^2] = \int_0^1 x^2(3\sqrt{x} - 3x^2)\,dx = 3\left(\frac{2}{7}x^{7/2} - \frac{1}{5}x^5\right)\Big|_0^1 = \frac{9}{35}$

$\mathrm{var}(X) = \mathrm{var}(Y) = E[X^2] - E^2[X] = \frac{9}{35} - \left(\frac{9}{20}\right)^2 = \frac{9 \cdot 80}{2800} - \frac{81 \cdot 7}{2800} = \frac{153}{2800} \approx 0.0546$

c) $E[XY] = \int_0^1 \int_{y^2}^{\sqrt{y}} 3xy\,dx\,dy = \int_0^1 3y \cdot \frac{1}{2}(y - y^4)\,dy = \frac{3}{2}\left(\frac{1}{3}y^3 - \frac{1}{6}y^6\right)\Big|_0^1 = \frac{1}{4}$

Thus $\rho_{X,Y} = \frac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{var}(X)\mathrm{var}(Y)}} = \frac{E[XY] - E[X]E[Y]}{\sqrt{\mathrm{var}(X)}\sqrt{\mathrm{var}(Y)}} = \frac{2800}{153} \cdot \left(\frac{1}{4} - \frac{81}{400}\right) = \frac{133}{153}$

$\approx 0.8693$