HOMEWORK ASSIGNMENT #5

Due Wed. Nov. 8, 2000 (in class)

1. Consider exponential random variable X with parameter λ.

   (a) Derive the Markov, Chebyshev, and Chernoff bounds for the quantity $P[X \geq a]$. State the range of a over which each bound is valid.

   (b) Compare the three bounds in (a) to the actual $P[X \geq a]$ using MATLAB. To do this generate one plot for each value of $\lambda = 2, 1, \frac{1}{4}$. Each plot should compare the four traces over the range $1 \leq a \leq 20$ and $0 \leq P \leq 1$ (*but don't plot the bounds outside of their valid domain!*).

   Hints: I found the following MATLAB *commands useful:* `linspace`, "`./`", "`.*`", `plot`, `axis`. *You can label your plots nicely using* `legend, num2str,title, xlabel, ylabel`.

2. Let $\mathbf{X} \in \mathbb{R}^3$ be a random (column) vector with mean $\boldsymbol{\mu} = (5, -5, 6)^t$ and covariance

$$\mathbf{K} = \begin{pmatrix} 5 & 2 & -1 \\ 2 & 5 & 0 \\ -1 & 0 & 4 \end{pmatrix}.$$

   Calculate the mean and variance of $Y = \mathbf{a}^t\mathbf{X} + b$ where $\mathbf{a} = (2, -1, 2)^t$ and $b = 5$.

3. Explain why none of the following matrices can be covariance matrices $\mathbf{K}_{XX}$ associated with a real random vector $\mathbf{X}$:

$$\underset{(a)}{\begin{pmatrix} 4 & 6 & 2 \\ 6 & 9 & 3 \\ 8 & 12 & 16 \end{pmatrix}} \quad \underset{(b)}{\begin{pmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -2 \end{pmatrix}} \quad \underset{(c)}{\begin{pmatrix} 6 & 1+j & 2 \\ 1-j & 5 & -1 \\ 2 & -1 & 6 \end{pmatrix}} \quad \underset{(d)}{\begin{pmatrix} 2 & -4 & 0 \\ -4 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}}$$

   (e) Would any be valid if $\mathbf{X}$ was complex-valued?

4. Prove that if $\{X_i\}_{i=1}^n$ are real jointly Gaussian random variables which are pairwise independent, then $\{X_i\}_{i=1}^n$ are independent.

5. Let $\{\mathbf{X}_i\}_{i=1}^n$ be mutually orthogonal complex random vectors. Show that

$$\mathrm{E}\left[\left\|\sum_{i=1}^{n} \mathbf{X}_i\right\|^2\right] = \sum_{i=1}^{n} \mathrm{E}\left[\|\mathbf{X}_i\|^2\right]$$

   where $\|\mathbf{x}\|^2 = \mathbf{x}^H\mathbf{x}$.

6. Consider again the transmission of a BPSK communication signal across a noisy channel. Let S denote the transmitted signal, with $P\{S = 1\} = P\{S = -1\} = 0.5$, let N denote real-valued Gaussian noise with zero mean and variance σ^2, and let R denote the received signal $R = S + N$. Assume signal and noise are independent. Given observation R, what is the MMSE estimate of S? *Hint: It involves* $\tanh(\cdot)$. Sketch $\hat{S}_{\mathrm{MMSE}}$ as a function of R (roughly, for some $0 < \sigma < 1$).