HOMEWORK ASSIGNMENT #4

Due Fri. Oct. 27, 2000 (in class)

1. Consider the random variables X, Y with joint pdf

$$f_{X,Y}(x,y) = \begin{cases} 2e^{-(x+2y)}, & x, y > 0 \\ 0, & \text{else.} \end{cases}$$

Let $Z = X + Y$ and $U = X/Y$. Find the joint pdf of Z, U.

2. Say $\boldsymbol{X}$ is a vector of N zero-mean jointly Gaussian random variables with (positive definite symmetric) covarance matrix $\boldsymbol{R}$. In other words, the joint pdf of $\boldsymbol{X}$ is:

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{1}{(2\pi)^{N/2}|\boldsymbol{R}|^{1/2}} e^{-\frac{1}{2}\boldsymbol{x}^t \boldsymbol{R}^{-1}\boldsymbol{x}}.$$

   (a) Say the elements of $\boldsymbol{X}$ are dependent. Find a matrix $\boldsymbol{W}$ so that $\boldsymbol{Y} = \boldsymbol{W}\boldsymbol{X}$ contains N independent random variables with equal variance σ^2. (Hint: consider the eigendecomposition of $\boldsymbol{R}$.)

   (b) Find $\mathbf{W}$ when $N = 2$, $\sigma = 1$, and

$$\mathbf{R} = \begin{pmatrix} 5 & -\sqrt{3} \\ -\sqrt{3} & 7 \end{pmatrix}.$$

3. Define the subset A as the region of the unit square included between the curves $y = x^2$ and $y = \sqrt{x}$. Consider the random variables X, Y with joint pdf

$$f_{XY}(x,y) = \begin{cases} 3, & (x,y) \in A \\ 0, & \text{else.} \end{cases}$$

(This should look familiar.)

   (a) Find $\mathrm{E}(X)$ and $\mathrm{E}(Y)$.

   (b) Find $\mathrm{var}(X)$ and $\mathrm{var}(Y)$.

   (c) Find $\rho_{X,Y}$.

4. Say that $X - Y^{10}$ is independent of Y. Prove $E(X|Y) = Y^{10} + c$, where c is some constant, and evaluate c.

5. Prove that if $E(X|Y,Z) = E(X|Y)$ then $E(XZ|Y) = E(X|Y)E(Z|Y)$.

6. Say that $\{X_i\}_{i=1}^n$ are independent identically distributed Bernoulli random variables with parameter p. Find a closed-form expression for the pmf of $Y = \sum_{i=1}^n X_i$. (Hint: Use the characteristic function.)

7. A popular expansion of the characteristic function is

$$\log \Phi_X(\omega) = \sum_{n=0}^{\infty} \mathcal{C}_{X,n} \frac{(j\omega)^n}{n!}$$

where $\mathcal{C}_{X,n}$ is called the n^{th} *cumulant* of X. Cumulants have many important uses in signal processing, communication theory, and many other fields. Find an expression for the n^{th} cumulant of $Y = \sum_{i=1}^{m} a_i X_i$, where $\{X_i\}$ are independent identically distributed and a_i are real constants.

8. Prove the *chain rule* for pdfs, which is useful in many problems:

$$f_{X_1,X_2,\ldots,X_n}(x_1,x_2,\ldots,x_n) =$$
$$f_{X_1|X_2,\ldots,X_n}(x_1|x_2,\ldots,x_n)\, f_{X_2|X_3,\ldots,X_n}(x_2|x_3,\ldots,x_n)\, f_{X_3|X_4,\ldots,X_n}(x_3|x_4,\ldots,x_n) \cdots f_{X_n}(x_n).$$