HOMEWORK ASSIGNMENT #3

Due Fri. Oct. 20, 2000 (in class)

1. Consider the a-centered clipper described bythe following transformation:

$$g(x) = \begin{cases} x - a, & x > a \\ 0, & |x| \leq a \\ x + a, & x < -a \end{cases}$$

   (a) Find the pdf of $Y = g(X)$.
   (b) Find the cdf of $Y = g(X)$.
   (c) What is the pdf of Y when X is discrete with $P\{X = 2a\} = 0.5$ and $P\{X = -2a\} = 0.5$?

2. The length of time, Z, and airplane can fly is given by $Z = \alpha X$ where X is the amount of fuel in its tank and $\alpha > 0$ is some constant of proportionality. Suppose a plane has two separate fuel tanks so that when one becomes empty the other one is connected automatically. Say that a plane takes off before its fuel tanks have been checked. Let X_1 be the amount of fuel in its first tank and X_2 the amount in the second, and let X_1 and X_2 be modelled as independent random variables uniformly distributed over the interval $[0, b]$.

   (a) Compute the pdf of Z, the maximum flying time of this plane.
   (b) If $b = 100$ (liters) and $\alpha = 0.1$ (hours/liter), what is the probability that the plane will fly more than five hours?

3. Consider the transmission of a BPSK communication signal across a noisy channel. Let S denote the transmitted signal, with $P\{S = 1\} = 0.5$ and $P\{S = -1\} = 0.5$, let N denote Gaussian noise with zero mean and variance σ^2, and R denote the observed signal:

$$R = S + N.$$

   We assume that S and N are independent random variables.

   (a) Find an expression for $f_{R|S}(r|s)$.
   (b) Find an expression for $f_R(r)$ and sketch it (roughly, for some $0 < \sigma < 1$).
   (c) Find an expression for $f_{S|R}(s|r)$.
   (d) Calculate $P\{S = 1|R = 1\}$ and $P\{S = -1|R = 1\}$ when $\sigma = 1$.

   Hint: Use $f_S(s) = 0.5\,\delta(s-1) + 0.5\,\delta(s+1)$, where

$$\delta(s-a) = 0 \qquad \forall s \neq a,$$
$$\int_{-\infty}^{\infty} \delta(s-a)ds = 1 \qquad \forall a$$
$$\int_{-\infty}^{\infty} g(s)\delta(s-a)ds = g(a) \qquad \forall a. \qquad \text{(the “sifting property”)}$$

4. Let H and S be two independent random variables, where H is zero-mean Gaussian and S is discrete with $P\{S = 1\} = 0.5$ and $P\{S = -1\} = 0.5$. Let $R = HS$.

   (a) Find the pdf of R.

   (b) Are R and S independent?

   (c) Are R and H independent?

   Note: The model above describes a BPSK signal transmitted over a "flat Rayleigh fading" communication channel.

5. Say X is an exponential random variable with parameter $\lambda = 1$. What function $g(\cdot)$ will make $Y = g(X)$ uniform on the interval $[0, 1)$? Hint: Its not difficult to guess the right g based on a sketch of f_X, but be sure to prove that it is the desired transformation.