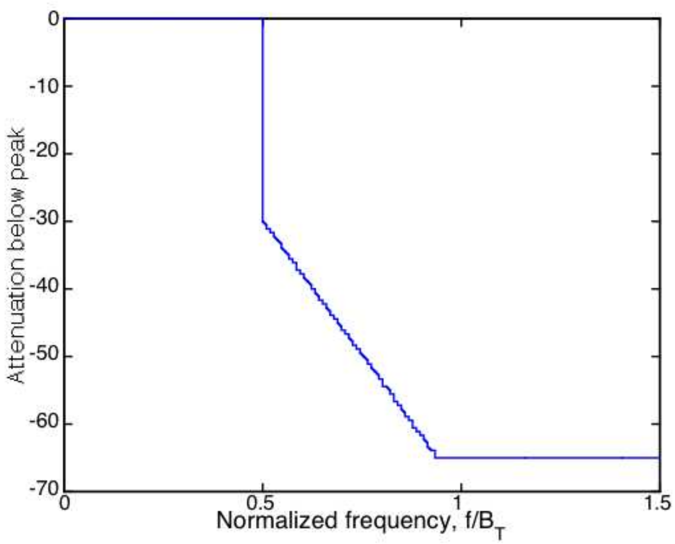
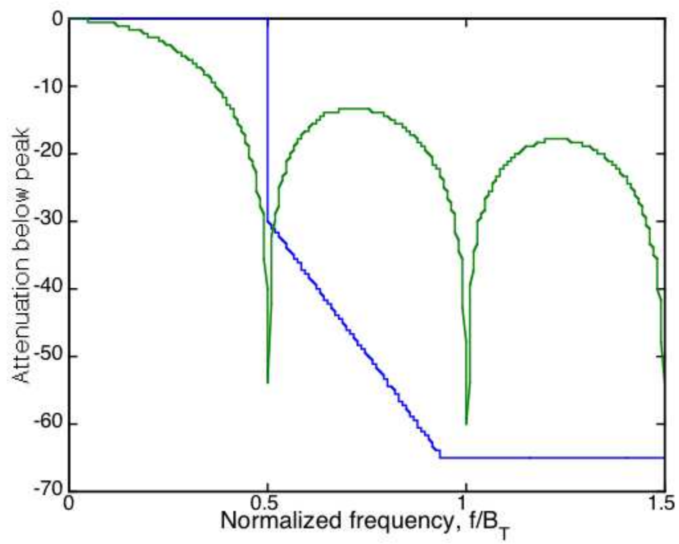
Spectrally Efficient Modulation [Ch. 16]:

- Out-of-band spectral content should be minimized.
- Often spectrum must fit into a "mask."
- The rectangular pulses we have assumed up until now have a sinc^2 energy spectrum: lots of spectral leakage!





Raised-cosine (RC) pulse family:

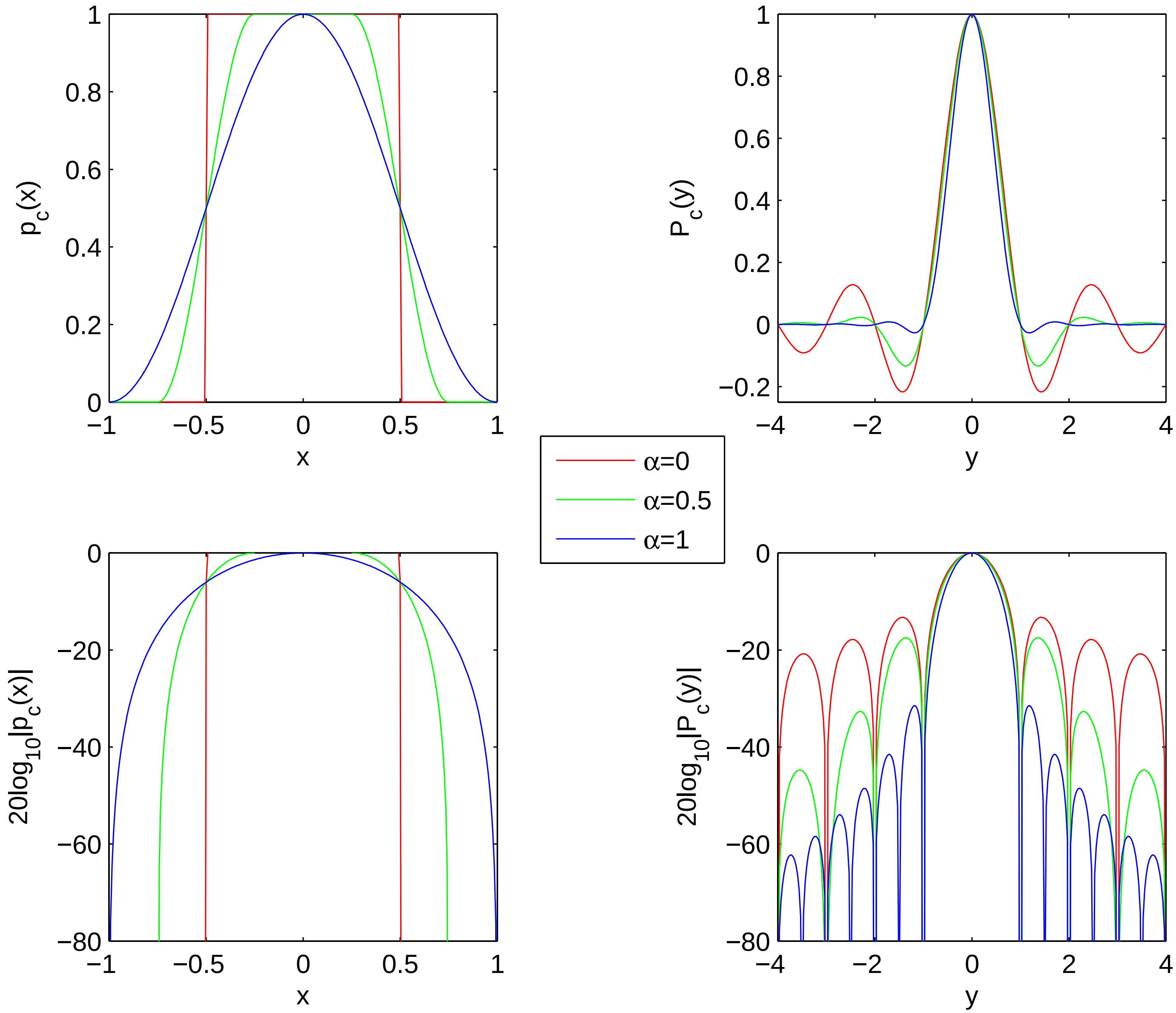
For parameter $\alpha \in [0, 1]$:

$$p_c(x) = \begin{cases} 1 & |x| \le \frac{(1-\alpha)}{2} \\ \cos^2\left(\frac{\pi}{2\alpha}\left(|x| - \frac{(1-\alpha)}{2}\right)\right) & \frac{(1-\alpha)}{2} \le |x| \le \frac{(1+\alpha)}{2} \\ 0 & \text{else} \end{cases}$$

$$P_c(y) = \mathcal{F}\{p_c(x)\} = \frac{\cos(\pi\alpha y)}{1-(2\alpha y)^2}\,\mathrm{sinc}(y),$$

where $\mathrm{sinc}(x) \triangleq \frac{\sin(\pi x)}{\pi x}$.

Note: x and y could denote either time or frequency!


$p_c(x)$
x
$P_c(y)$
y
α=0
α=0.5
α=1
$20\log_{10}|p_c(x)|$
x
$20\log_{10}|P_c(y)|$
y

Comments:

- Sidelobes of $P_c(y)$ get smaller as α gets larger.
- $p_c(x)\big|_{\alpha=0}$ = rectangular pulse of width 1.
- $p_c(x)\big|_{\alpha=1} = \begin{cases} \frac{1}{2} + \frac{1}{2}\cos(\pi x) & |x| \leq 1 \\ 0 & \text{else} \end{cases}$
- Fitz calls these "squared-cosine pulses," and presents $p_c(x/T_z)$ for scale parameter T_z.
- $P_c(y) = 0$ for non-zero integers y.
- $\int_{-\infty}^{\infty} P_c(y)dy = \int_{-\infty}^{\infty} p_c(x)dx = 1$

Square-root raised-cosine (SRRC) pulses:

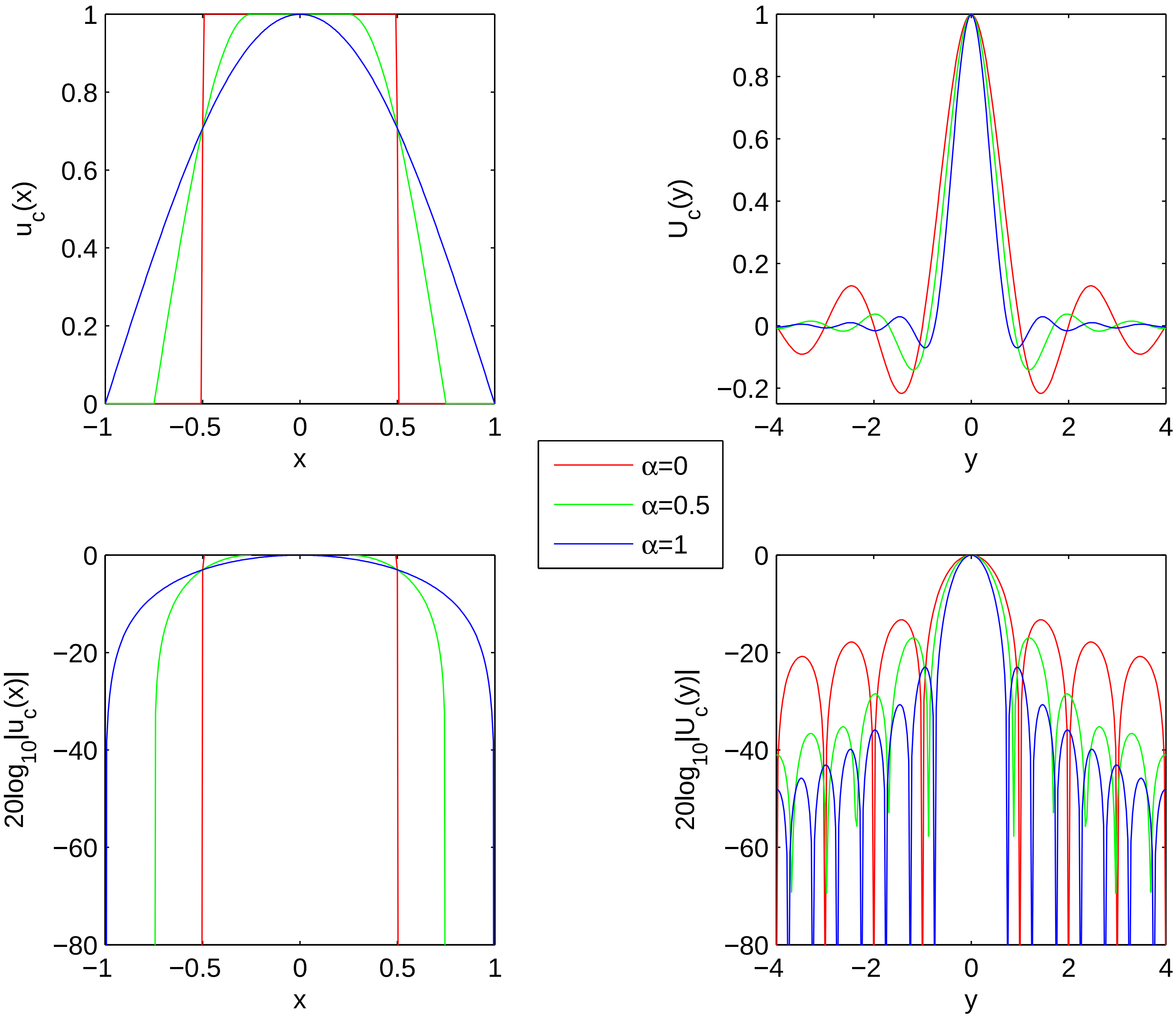
For parameter $\alpha \in [0, 1]$:

$$u_c(x) = \begin{cases} 1 & |x| \le \frac{(1-\alpha)}{2} \\ \cos\left(\frac{\pi}{2\alpha}\left(|x| - \frac{(1-\alpha)}{2}\right)\right) & \frac{(1-\alpha)}{2} \le |x| \le \frac{(1+\alpha)}{2} \\ 0 & \text{else} \end{cases}$$

$$U_c(y) = \frac{(1-\alpha)\,\mathrm{sinc}(y(1-\alpha))}{1-(4\alpha y)^2} + \frac{4\alpha\cos(\pi y(1+\alpha))}{\pi(1-(4\alpha y)^2)}$$

Main points:

- $u_c(x) = \sqrt{p_c(x)}$ ($\Rightarrow$ unit energy: $\int_{-\infty}^{\infty} |u_c(x)|^2 dx = 1$)
- $\int_{-\infty}^{\infty} U_c(y) U_c(y-n) dy = \delta_n$ (the Kronecker delta)
- Fitz calls them "cosine pulses".



1
0.8
0.6
0.4
0.2
0
$u_c(x)$
−1
−0.5
0
0.5
1
x
−0.2
$U_c(y)$
−4
−2
0
2
4
y
α=0
α=0.5
α=1
−20
−40
−60
−80
$20\log_{10}|u_c(x)|$
$20\log_{10}|U_c(y)|$

Spectral shaping for OFDM:

$$X_z(t) = \sum_{l=1}^{L} D_z^{(l)} \sqrt{\frac{E_b}{T_p}} u_s(t) e^{j2\pi(\frac{2l-L-1}{2T_p})t}$$

where previously $u_s(t) = 1_{[0,T_p]}(t)$, but now $u_s(t)$ is general.

Orthogonality condition:

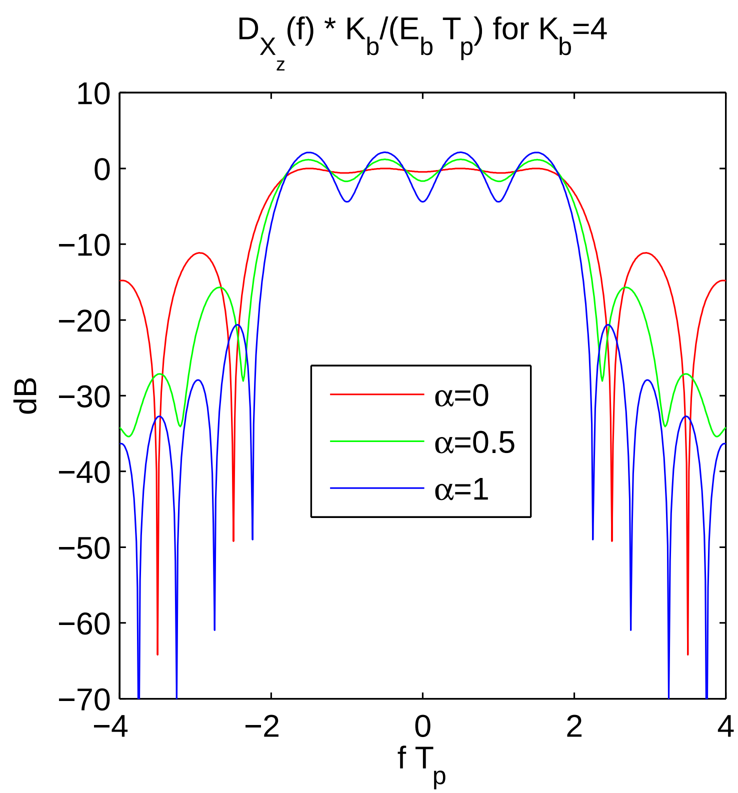
$$\begin{aligned} 0 &= \mathrm{Re}\left[D_z^{(l)} D_z^{(k)*} \int_{-\infty}^{\infty} |u_s(t)|^2 e^{-j2\pi(\frac{k-l}{T_p})t} dt\right] \quad \forall k \neq l \\ &= \mathrm{Re}\left[D_z^{(l)} D_z^{(k)*} P_s(\tfrac{k-l}{T_p})\right] \text{ where } \; p_s(t) \triangleq |u_s(t)|^2 \end{aligned}$$

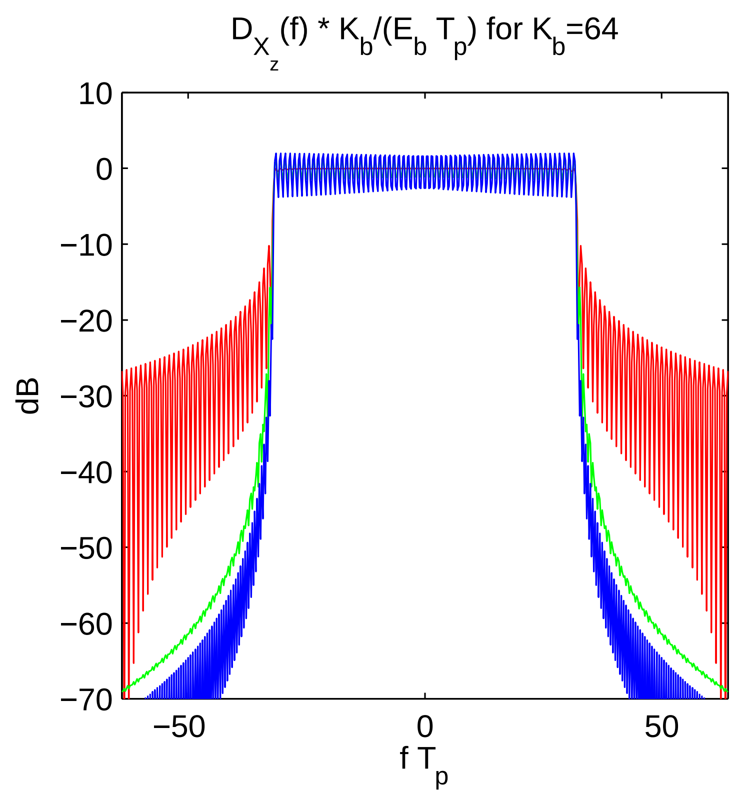
Choosing $u_s(t)$ as a SRRC pulse guarantees orthogonality:

$$u_s(t) = u_c(\tfrac{t}{T_p}) \;\Rightarrow\; P_s(\tfrac{k-l}{T_p}) = T_p P_c(k-l) = 0 \;\; \forall k \neq l.$$

Observations:

- Larger α leads to quicker sidelobe decay.
- Larger K_b makes pulse-shaping more effective.





Spectral shaping for linear stream modulation:

$$X_z(t) = \sum_{l=1}^{K_b} D_z^{(l)} \sqrt{E_b}\, u\big(t-(l-1)T\big)$$

Orthogonality condition:

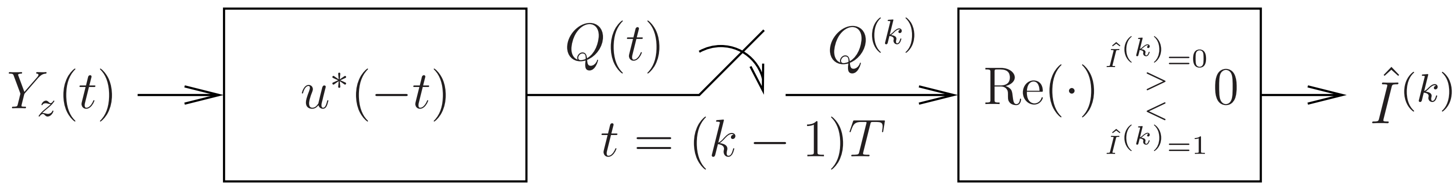
$$0 \;=\; \mathrm{Re}\Bigg[D_z^{(l)} D_z^{(k)*} \underbrace{\int_{-\infty}^{\infty} u(t-lT)u^*(t-kT)dt}_{\triangleq\, V_u\big((k-l)T\big)}\Bigg] \quad \forall k \neq l$$

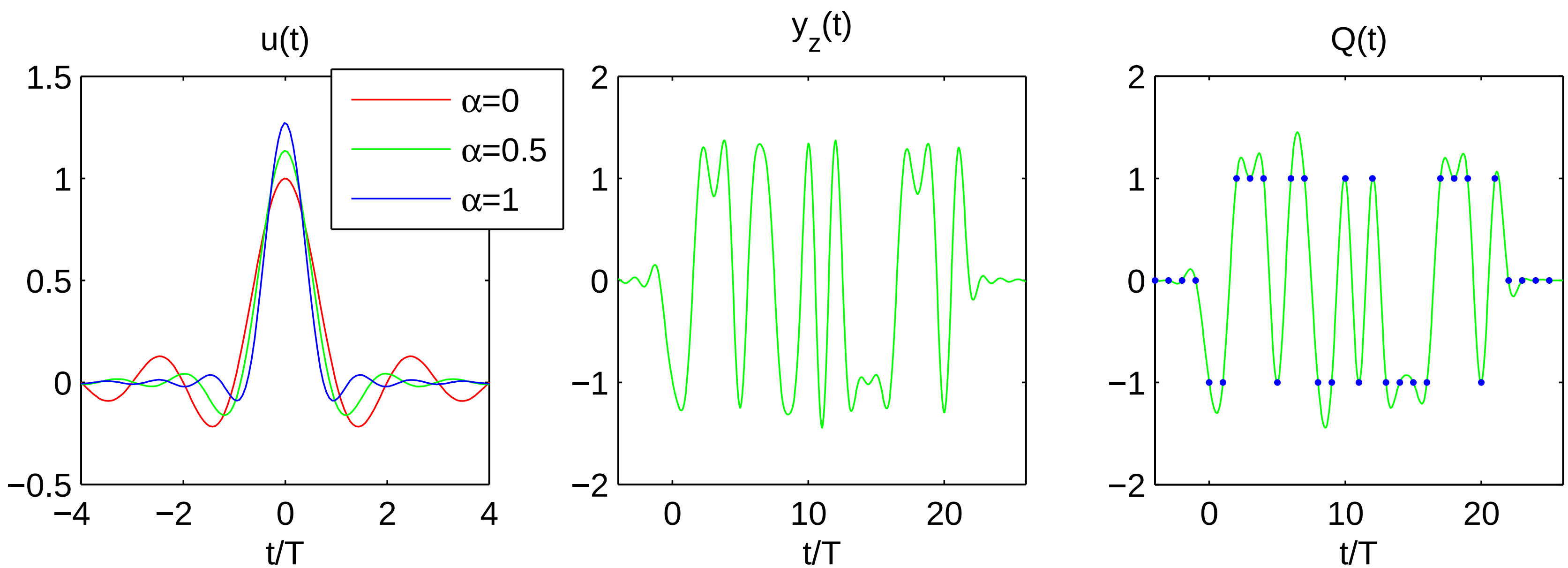
To guarantee orthogonality, use "y-domain" SRRC pulse:

$$u(t) = \tfrac{1}{\sqrt{T}} U_c(\tfrac{t}{T}) \quad \Rightarrow \quad V_u\big((k-l)T\big) = \delta_{k-l}.$$

Often called the "Nyquist criterion for zero ISI."

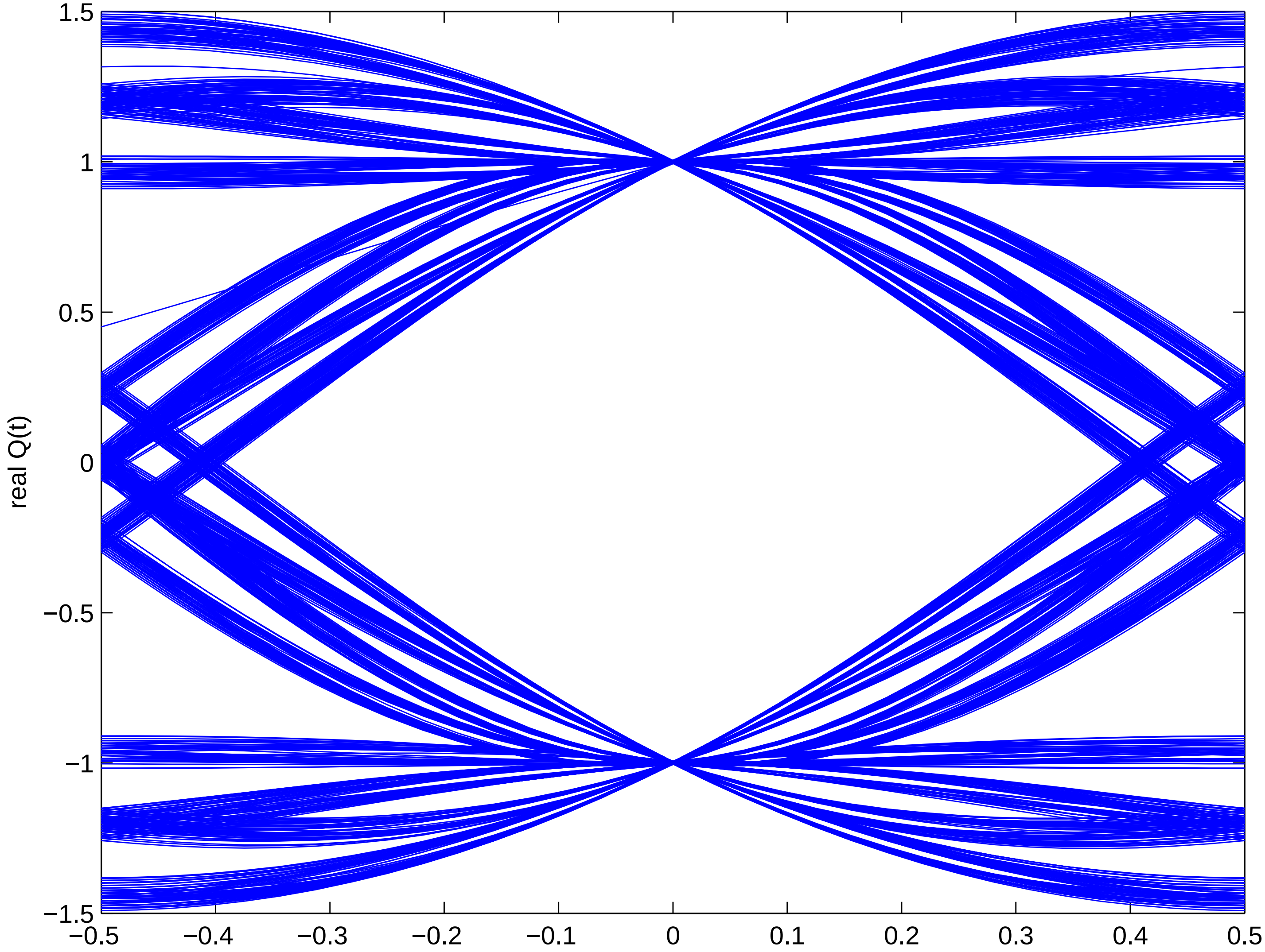
ML binary stream demodulation:



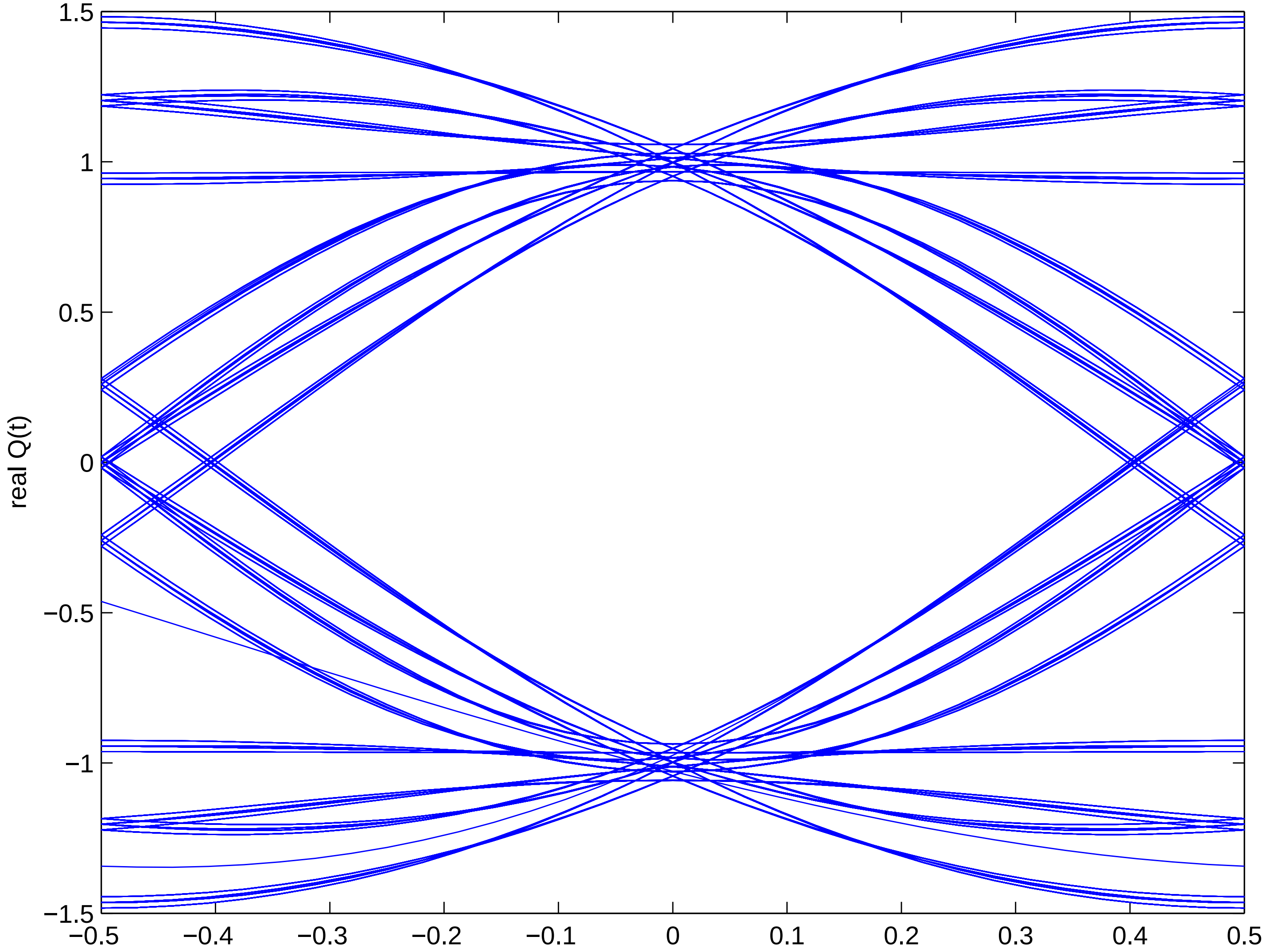


Note: non-causal (for simplicity). To make causal, shift pulse forward $T_u/2$ seconds and set to zero for $t \notin [0, T_u]$.

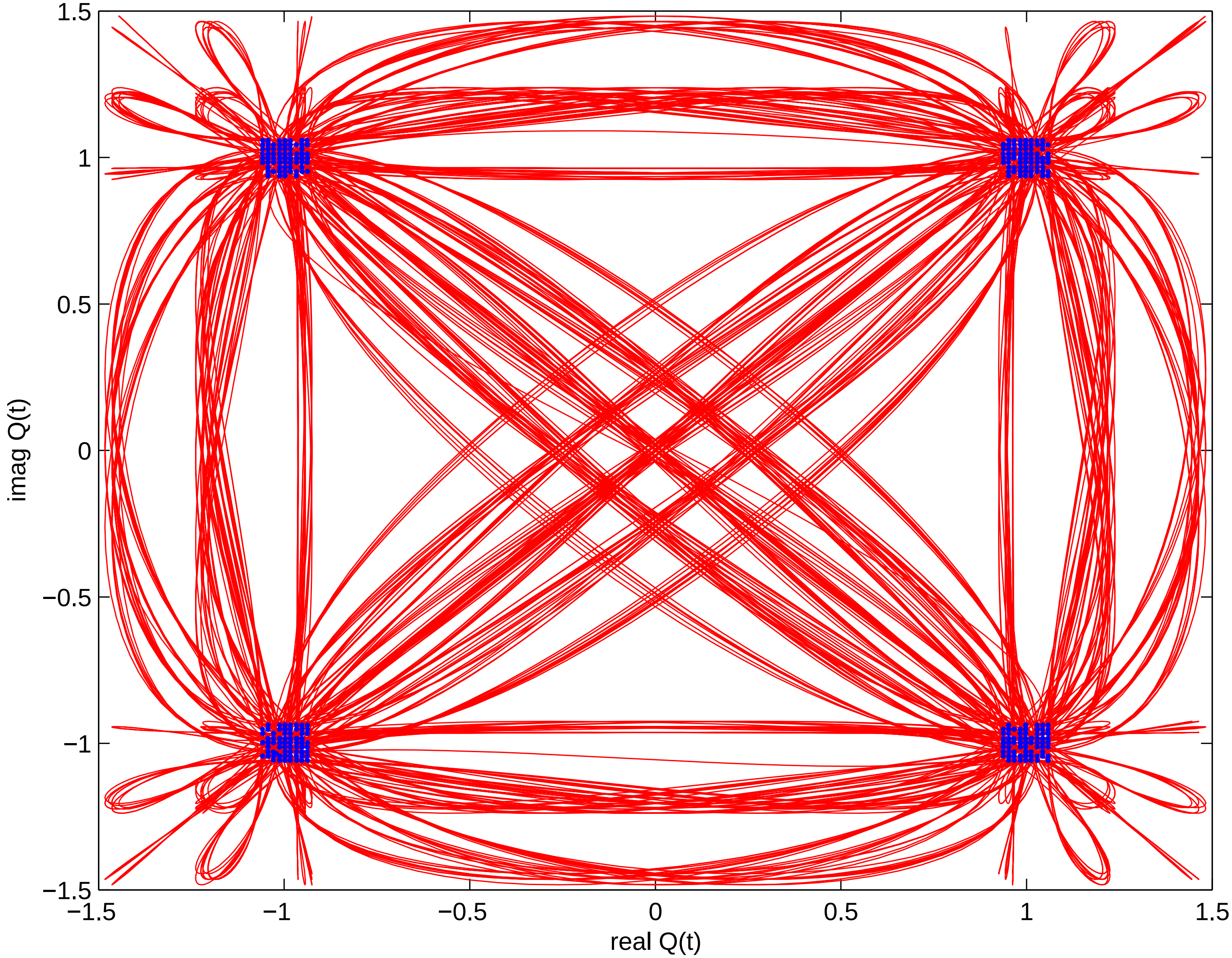
Eye diagram (BPSK, $\alpha = 0.5$, $T_u = 8T$):



Eye diagram (BPSK, $\alpha = 0.5$, $T_u = 3T$):



Vector diagram (QPSK, $\alpha = 0.5$, $T_u = 3T$):



Scatter plot (QPSK, $\alpha = 0.5$, $T_u = 3T$):