Managing Complexity [Ch. 15]:

Problem:

- Demodulating K_b bits generally requires $\mathcal{O}(2^{K_b})$ complexity.
- For send a fixed bit rate, we need $\mathcal{O}(K_b)$ complexity.

Insights:

- MPSK only needed one filter for K_b bits.
  $\rightsquigarrow$ linear modulation.
- Gray-coded QPSK demodulated each bit in parallel.
  $\rightsquigarrow$ orthogonal modulation.

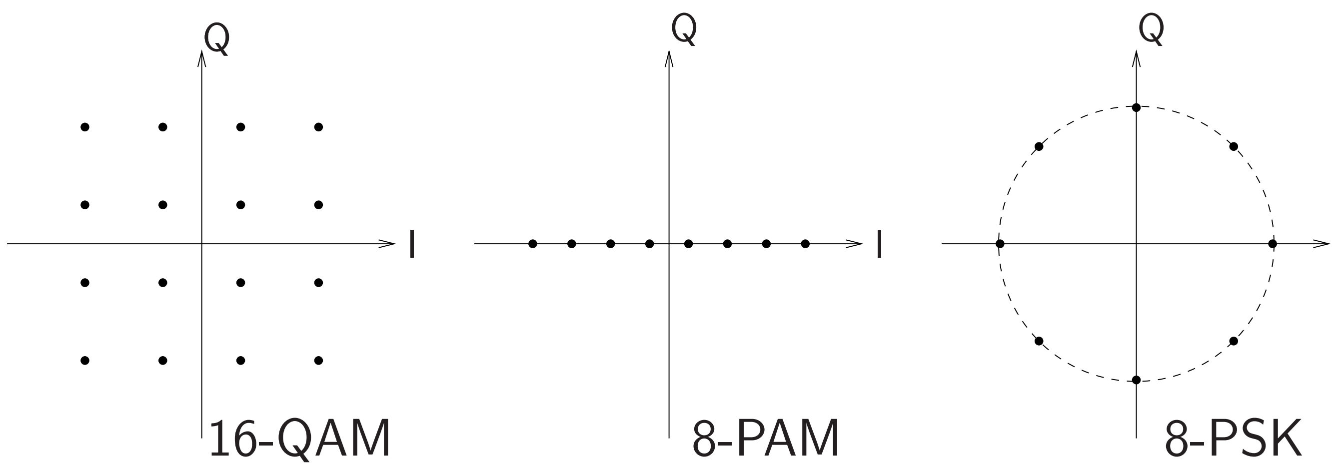
Linear Modulation:

$$\underline{I} = i \quad \Rightarrow \quad x_i(t) = d_i \sqrt{E_b} u(t)$$

Normalization so that average energy per bit $= E_b$:

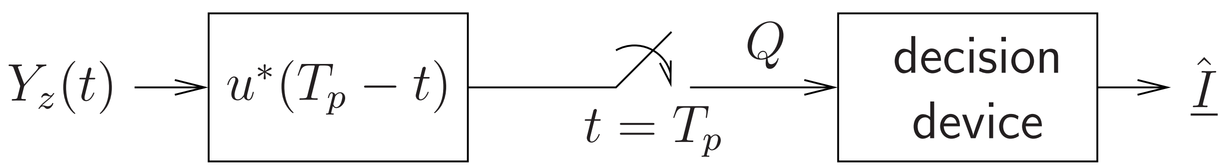
- $\int_{-\infty}^{\infty} |u(t)|^2 dt = 1$
- $\sum_{i=0}^{M-1} |d_i|^2 \pi_i = K_b$ (so that avg word energy $= K_b E_b$)

Example constellations:

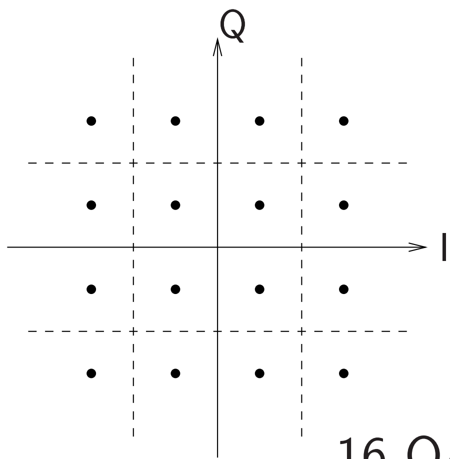


MLWD for linear modulation (under equal priors):

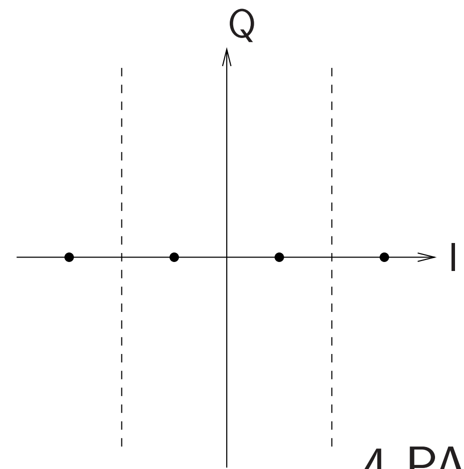
$$
\begin{aligned}
E_i &= |d_i|^2 E_b \\
V_i(T_p) &= \int_0^{T_p} Y_z(t) x_i^*(t)\,dt \;=\; d_i^* \sqrt{E_b} \underbrace{\int_0^{T_p} Y_z(t) u^*(t)\,dt}_{Q} \\
\underline{\hat{I}} &= \arg\max_i \operatorname{Re}\{V_i(T_p)\} - E_i/2 \\
&= \arg\max_i \sqrt{E_b} \operatorname{Re}\{d_i^* Q\} - |d_i|^2 E_b/2 \\
&= \arg\max_i -\left|Q - d_i\sqrt{E_b}\right|^2 \\
&= \arg\min_i \left|Q - d_i\sqrt{E_b}\right|^2 \quad (\text{“minimum distance decoder”})
\end{aligned}
$$



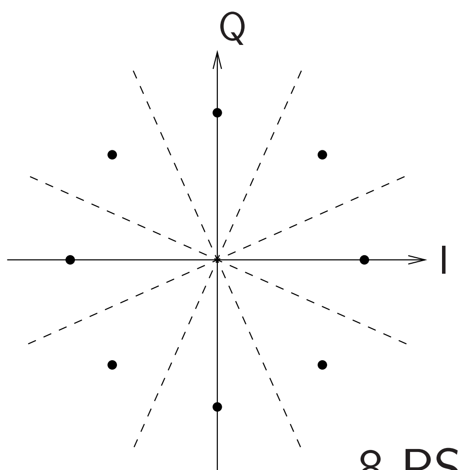
Example decision regions:



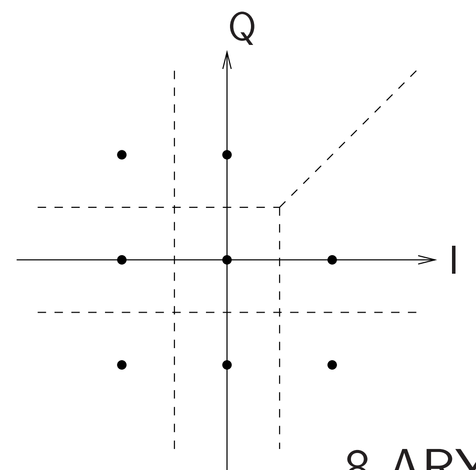
16-QAM



4-PAM



8-PSK



8-ARY ?

Properties of MLWD with linear modulation:

- Only a single filter required.
- Decision $\hat{\underline{I}} = i$ inferred when $Q \in$ decision region A_i.
- $\Pr\big(\hat{\underline{I}} \neq i | \underline{I} = i\big) = \Pr\big(Q \notin A_i | \underline{I} = i\big)$

Notice that

$$\begin{aligned} Q\Big|_{\underline{I}=i} &= \int_0^{T_p} \Big(d_i \sqrt{E_b} u(t) + W_z(t)\Big) u^*(t) dt \\ &= d_i \sqrt{E_b} + N_z(T_p) \\ \sigma^2_{N_z(T_p)} &= \int_{-\infty}^{\infty} N_0 |U(f)|^2 df \;=\; N_0 \int_{-\infty}^{\infty} |u(t)|^2 dt \;=\; N_0, \end{aligned}$$

and so $Q\Big|_{\underline{I}=i} \sim \mathcal{CN}(d_i \sqrt{E_b}, N_0)$.

Exact WEP analysis:

$$\text{WEP} = \frac{1}{M}\sum_{i=0}^{M-1} \Pr(Q \notin A_i | \underline{I} = i),$$

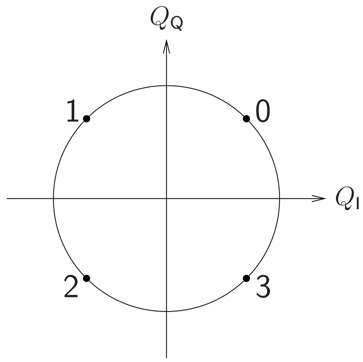
where

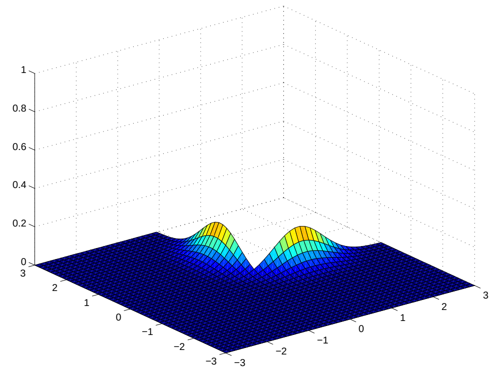
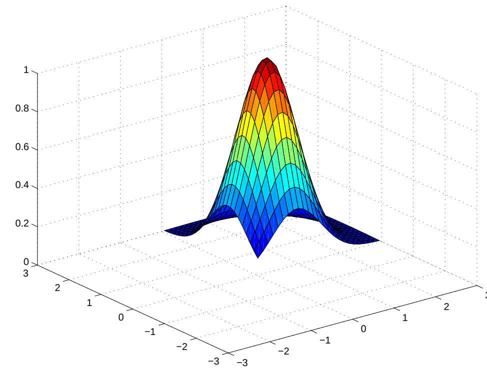
$$\begin{aligned} \Pr(Q \notin A_i | \underline{I} = i) &= 1 - \Pr(Q \in A_i | \underline{I} = i) \\ &= 1 - \int_{A_i} f_{Q|\underline{I}}(q|i)\, dq, \end{aligned}$$

so we need to integrate the pdf of $Q\big|_{\underline{I}=i} \sim \mathcal{CN}(d_i\sqrt{E_b}, N_0)$ over the decision region A_i.

Exact WEP for QPSK:

$$\sum_{i=0}^{M-1} |d_i|^2 \pi_i = K_b \quad \Rightarrow \quad |d_i| = \sqrt{2} \quad \Rightarrow \quad d_{i\mathsf{Q}}, d_{i\mathsf{I}} = \pm 1.$$



$$\begin{aligned}
&\Pr(Q \notin A_0 | \underline{I} = 0) \\
&= 1 - \int_{A_0} f_{Q|\underline{I}}(q|0) dq \\
&= 1 - \int_0^\infty \int_0^\infty \frac{1}{\pi N_0} e^{\left[-\frac{(q_{\mathsf{Q}} - \sqrt{E_b})^2 + (q_{\mathsf{I}} - \sqrt{E_b})^2}{N_0}\right]} dq_{\mathsf{Q}} dq_{\mathsf{I}}.
\end{aligned}$$

Can decouple this double integral...

$$
\begin{aligned}
&\Pr(Q \notin A_0 | \underline{I} = 0) \\
&= 1 - \int_0^{\infty} \frac{1}{\sqrt{2\pi(N_0/2)}} e^{-\frac{(q_{\mathsf{Q}} - \sqrt{E_b})^2}{2(N_0/2)}} dq_{\mathsf{Q}} \int_0^{\infty} \frac{1}{\sqrt{2\pi(N_0/2)}} e^{-\frac{(q_{\mathsf{I}} - \sqrt{E_b})^2}{2(N_0/2)}} dq_{\mathsf{I}} \\
&= 1 - \Pr\{Q_{\mathsf{Q}} > 0\}^2 \;=\; 1 - \big(1 - F_{Q_{\mathsf{Q}}}(0)\big)^2 \\
&= 1 - \left[\frac{1}{2} - \frac{1}{2}\operatorname{erf}\left(\frac{0 - \sqrt{E_b}}{\sqrt{2}\sqrt{N_0/2}}\right)\right]^2 \\
&= 1 - \frac{1}{4}\left[1 + \operatorname{erf}\left(\sqrt{\frac{E_b}{N_0}}\right)\right]^2 \;=\; 1 - \frac{1}{4}\left[2 - \operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)\right]^2 \\
&= \operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right) - \frac{1}{4}\operatorname{erfc}^2\left(\sqrt{\frac{E_b}{N_0}}\right) \\
&= \text{WEP (by symmetry).}
\end{aligned}
$$

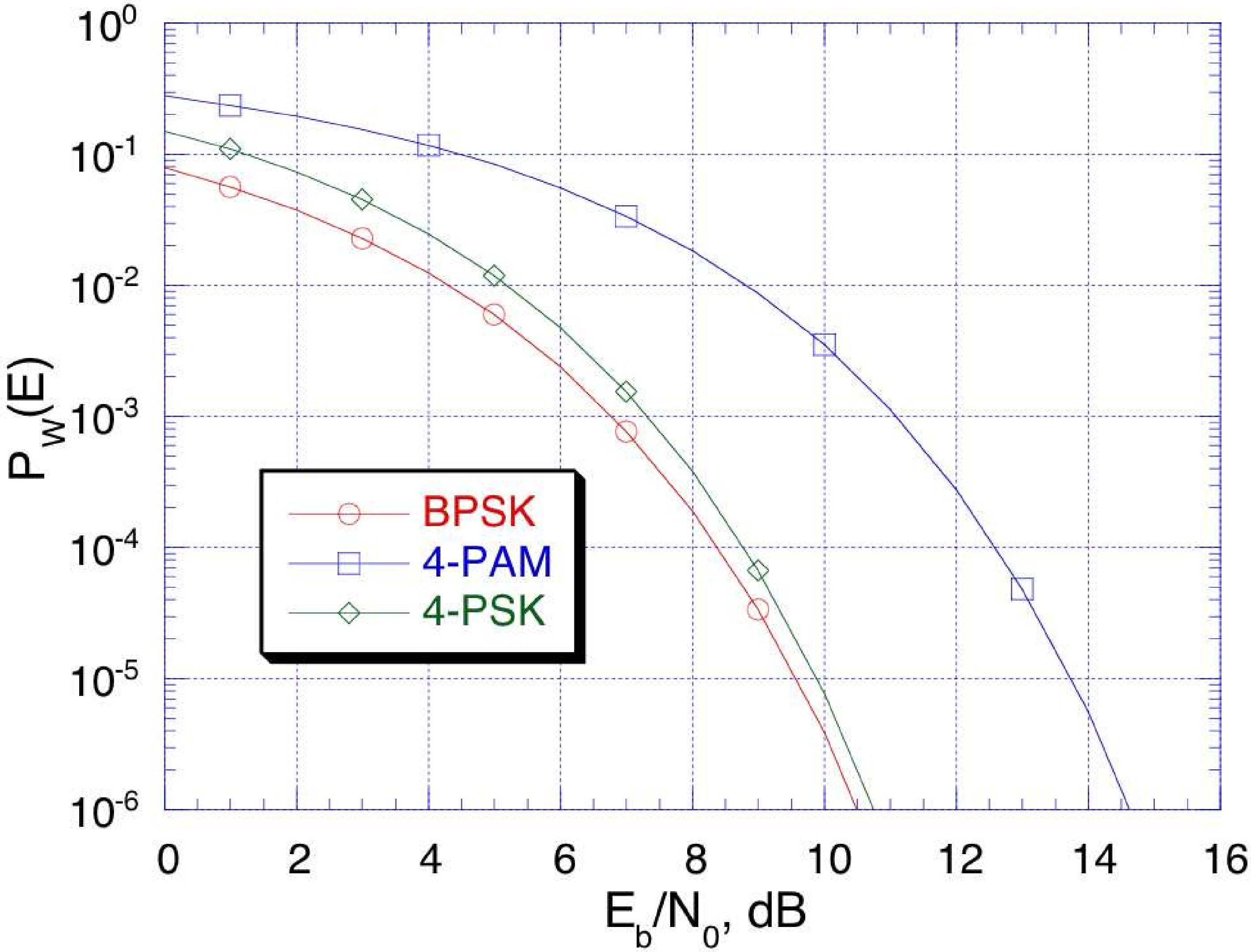
Exact WEP for some simple linear modulations:

BPSK: $\frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$

QPSK: $\mathrm{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right) - \frac{1}{4}\,\mathrm{erfc}^2\left(\sqrt{\frac{E_b}{N_0}}\right)$

4-PAM: $\frac{3}{4}\,\mathrm{erfc}\left(\sqrt{\frac{2E_b}{5N_0}}\right)$

Exact WEP for some simple linear modulations:



Spectral characteristics of linear modulation:

Since $x_i(t) = d_i\sqrt{E_b}u(t)$, we find that

$$G_{X_i}(f) = |d_i|^2 E_b G_U(f),$$

and hence

$$\begin{aligned} D_{X_z}(f) &= \frac{1}{K_b}\sum_{i=0}^{M-1} \pi_i G_{X_i}(f) \\ &= \frac{E_b}{K_b} G_U(f) \underbrace{\sum_{i=0}^{M-1} |d_i|^2 \pi_i}_{K_b} \\ &= E_b G_U(f) \end{aligned}$$

Note: Energy spectrum depends only on pulse shape $u(t)$.

Summary of linear modulation:

- MLWD: Match-filter via pulse shape $u(t)$ and quantize output Q to nearest constellation point.
- Depending on the decision regions, could still be $\mathcal{O}(2^{K_b})$ complexity.
- Possible to derive exact WEP by integrating Gaussian pdfs over the decision regions.
- Energy spectrum depends only on pulse shape $u(t)$. (Later, we discuss "good" choices for $u(t)$.)

Orthogonal Modulation:

Decoupled ML Bit Decisions:

- Recall that this happened with Gray-coded QPSK.
- Motivation: gives $\mathcal{O}(K_b)$ complexity MLWD. (Recall that we need $\mathcal{O}(K_b)$ complexity demodulation to decode a constant-bit-rate stream.)
- Question: Exactly when can MLWD be implemented using decoupled decisions on each bit?

Say that $\underline{I} = \left[I^{(1)}, I^{(2)}, \ldots, I^{(K_b)}\right]$. Furthermore, say that

$$\underline{I} = i \quad \Leftrightarrow \quad \underline{I} = [m_1, m_2, \ldots, m_{K_b}].$$

Claim: If the ML metrics $\{T_i\}$ can be written in the form

$$T_i \; = \; \sum_{k=1}^{K_b} T_{m_k}^{(k)},$$

then MLWD is implementable with K_b decoupled decisions.

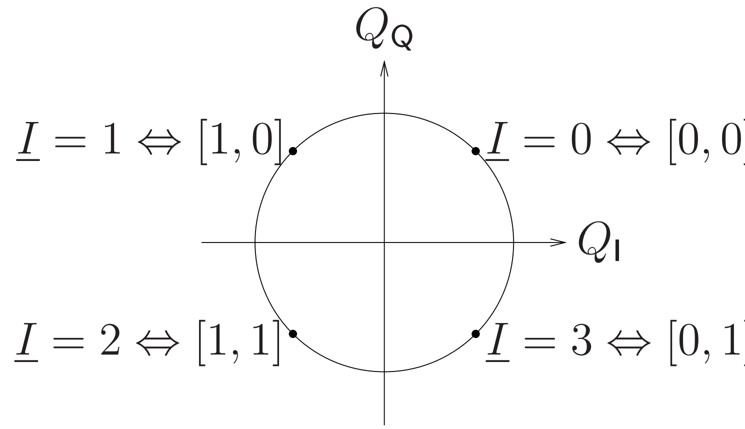
I.e., T_i is maximized by maximizing each $T_{m_k}^{(k)}$ separately:

$$\hat{\underline{I}} = \arg\max_i T_i \quad \Leftrightarrow \quad \hat{\underline{I}} = \left[\arg\max_{m_1} T_{m_1}^{(1)}, \ldots, \arg\max_{m_{K_b}} T_{m_{K_b}}^{(K_b)}\right]$$

Example: Gray Coded QPSK:

Bit-to-symbol mapping

$$d_i = d_{m_1} + j d_{m_2}$$



implies that

$$\begin{aligned} T_i &= \sqrt{E_b}\,\mathrm{Re}\{d_i^* Q\} - E_b \qquad \text{(from p. 3)} \\ &= \sqrt{E_b}\,\mathrm{Re}\{(d_{m_1} - j d_{m_2})(Q_{\mathsf{I}} + j Q_{\mathsf{Q}})\} - E_b \\ &= \underbrace{\sqrt{E_b}\, d_{m_1} Q_{\mathsf{I}} - \frac{E_b}{2}}_{T_{m_1}^{(1)}} + \underbrace{\sqrt{E_b}\, d_{m_2} Q_{\mathsf{Q}} - \frac{E_b}{2}}_{T_{m_2}^{(2)}}. \end{aligned}$$

Generic Orthogonal Modulation:

The k^{th} bit chooses between the waveforms in $\{x_0^{(k)}(t), x_1^{(k)}(t)\}$, where $\{x_0^{(k)}(t), x_1^{(k)}(t)\} \perp \{x_0^{(l)}(t), x_1^{(l)}(t)\}$ for $k \neq l$, and then the sum of waveforms for bits $k \in \{1, \ldots, K_b\}$ is transmitted.

In other words, if $\underline{I} = i = [m_1, m_2, \ldots, m_{K_b}]$, then

$$\begin{aligned} x_i(t) &= \sum_{k=1}^{K_b} x_{m_k}^{(k)}(t), \quad \text{where} \\ 0 &= \mathrm{Re} \int_{-\infty}^{\infty} x_{m_k}^{(k)}(t) x_{m_l}^{(l)*}(t) dt \quad \forall m_k, m_l, k \neq l. \end{aligned}$$

Can show that this guarantees decoupled ML metrics. . .

$$
\begin{aligned}
T_i &= \mathrm{Re} \int_0^{T_p} Y_z(t) x_i^*(t) dt - \frac{1}{2} \int_0^{T_p} |x_i(t)|^2 dt \\
&= \mathrm{Re} \int_0^{T_p} Y_z(t) \left(\sum_{k=1}^{K_b} x_{m_k}^{(k)*}(t) \right) dt - \frac{1}{2} \int_0^{T_p} \left| \sum_{k=1}^{K_b} x_{m_k}^{(k)}(t) \right|^2 dt \\
&= \sum_{k=1}^{K_b} \mathrm{Re} \int_0^{T_p} Y_z(t) x_{m_k}^{(k)*}(t) dt - \frac{1}{2} \sum_{k=1}^{K_b} \sum_{l=1}^{K_b} \int_0^{T_p} x_{m_k}^{(k)}(t) x_{m_l}^{(l)*}(t) dt \\
&= \sum_{k=1}^{K_b} \underbrace{\left[\mathrm{Re} \int_0^{T_p} Y_z(t) x_{m_k}^{(k)*}(t) dt - \frac{1}{2} \int_0^{T_p} |x_{m_k}^{(k)}(t)|^2 dt \right]}_{T_{m_k}^{(k)}}
\end{aligned}
$$

Note: This specifies exactly how to generate $\{T_{m_k}^{(k)}\}$ for orthogonal modulations.

WEP for orthogonal modulation:

We just saw that M-ary MLWD decouples into K_b MLBDs. For the k^{th} MLBD, we know that

$$\text{BEP}^{(k)} = \frac{1}{2}\,\text{erfc}\left(\sqrt{\frac{\Delta_E^{(k)}(1,0)}{4N_0}}\right) \geq \frac{1}{2}\,\text{erfc}\left(\sqrt{\frac{E_b}{N_o}}\right),$$

where $\Delta_E^{(k)}(1,0) = \int_0^{T_p} |x_1^{(k)} - x_0^{(k)}|^2 dt$.

Since the probability of a correct word decision equals the probability of K_b simultaneously correct bit decisions,

$$\text{WEP} = 1 - \prod_{k=1}^{K_b}\left(1 - \text{BEP}^{(k)}\right) \geq 1 - \left[1 - \tfrac{1}{2}\,\text{erfc}\left(\sqrt{\tfrac{E_b}{N_o}}\right)\right]^{K_b}$$

Summary of M-ary orthogonal modulation:

- MLWD decouples into K_b MLBDs.
- MLWD implementable with $\mathcal{O}(K_b)$ complexity.
- WEP analysis reduces to BEP analysis.
- Performance is, at best, equal to binary antipodal signaling, which was far from Shannon's bound!

Can construct orthogonal waveforms by time-division, frequency-division, or "code-division"...

Example 1: Orthogonal Frequency Division Multiplexing:

For the case of *one bit per subcarrier*,

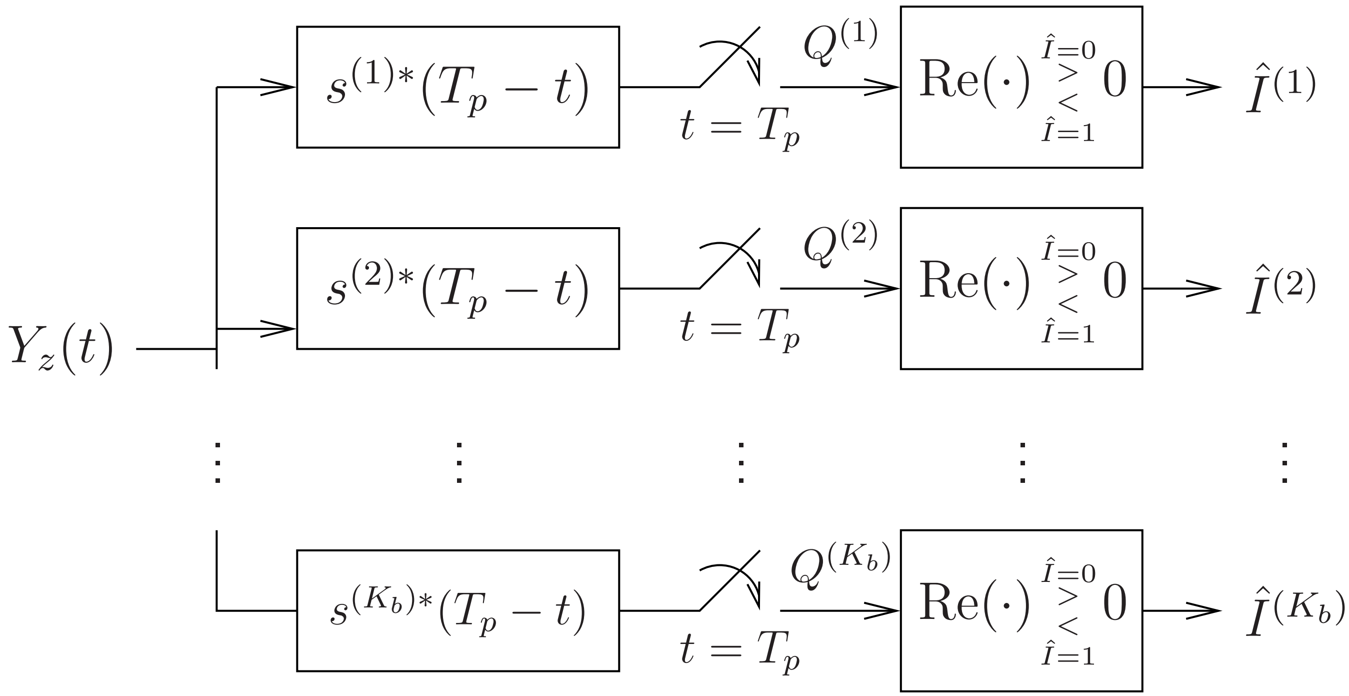
$$s^{(l)}(t) = \begin{cases} \sqrt{\frac{1}{T_p}} \exp\big(j2\pi f_d(2l - K_b - 1)t\big) & t \in [0, T_p] \\ 0 & t \notin [0, T_p] \end{cases}$$

$$X_z(t) = \sum_{l=1}^{K_b} \underbrace{D_z^{(l)} \sqrt{E_b} s^{(l)}(t)}_{x_{I^{(l)}}^{(l)}(t)}$$

using BPSK: $D_z^{(l)} = a(I^{(l)})$, i.e., $a(0) = 1,\ a(1) = -1$.

For orthogonality (i.e., $\mathrm{Re} \int_{-\infty}^{\infty} x_{m_l}^{(l)}(t) x_{m_k}^{(k)*}(t) = 0$), generally need $f_d = \frac{1}{2T_p}$, though $f_d = \frac{1}{4T_p}$ suffices for real-valued constellations. Still, we focus on $f_d = \frac{1}{2T_p}$.

Binary OFDM demodulator:



BEP identical to that of BPSK. Spectral efficiency is

$$\left.\begin{array}{l} W_b = \frac{K_b}{T_p} \text{ bits/sec,} \\ B_T \approx 2 f_d K_b, \;\; f_d = \frac{1}{2T_p}\text{Hz} \end{array}\right\} \Rightarrow \eta_B \approx 1 \text{ bit/sec/Hz}$$

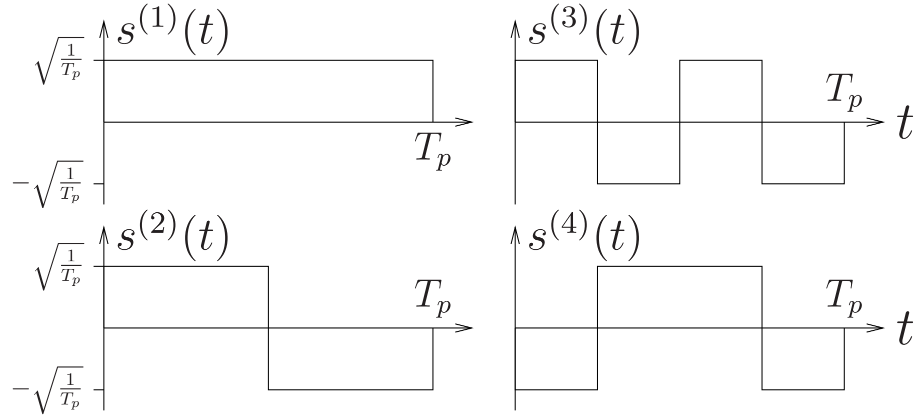
Example 2: Orthogonal Code Division Multiplexing:

For the case of *one bit per spreading waveform*,

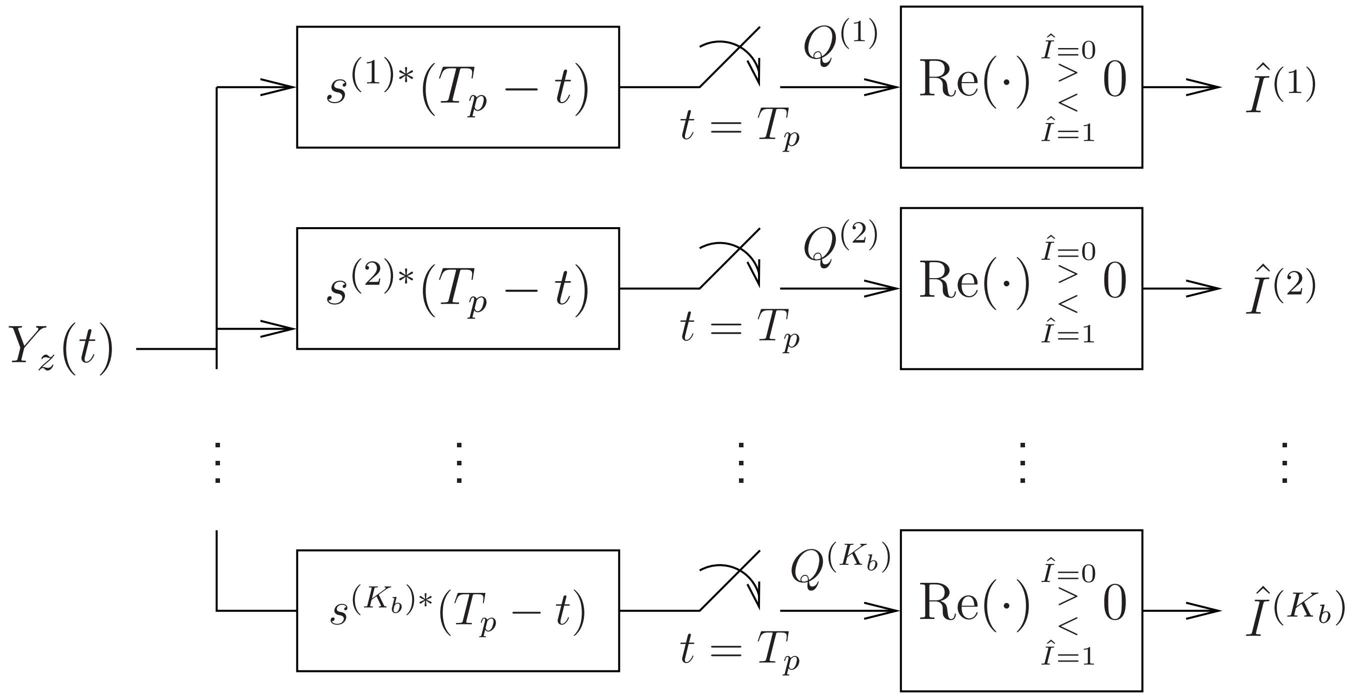
$$X_z(t) \;=\; \sum_{l=1}^{K_b} D_z^{(l)} \sqrt{E_b}\, s^{(l)}(t)$$

using BPSK $D_z^{(l)} = a(I^{(l)})$, i.e., $a(0) = 1,\ a(1) = -1.$ The spreading waveforms $\{s^{(l)}(t)\}_{l=1}^{K_b}$ are orthonormal on $[0, T_p]$:

Walsh codes:



Binary OCDM demodulator:



BEP identical to that of BPSK. Spectral efficiency is

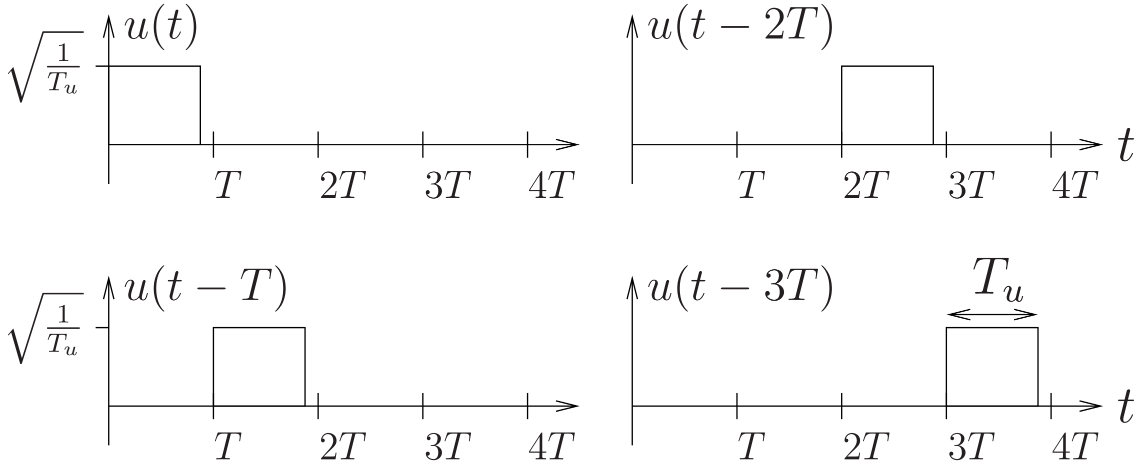
$$\left.\begin{array}{l} W_b = \frac{K_b}{T_p} \text{ bits/sec,} \\ B_T \geq \frac{K_b}{T_p} \text{ Hz} \end{array}\right\} \Rightarrow \eta_B \leq 1 \text{ bit/sec/Hz}$$

Example 3: Binary Stream Modulation:

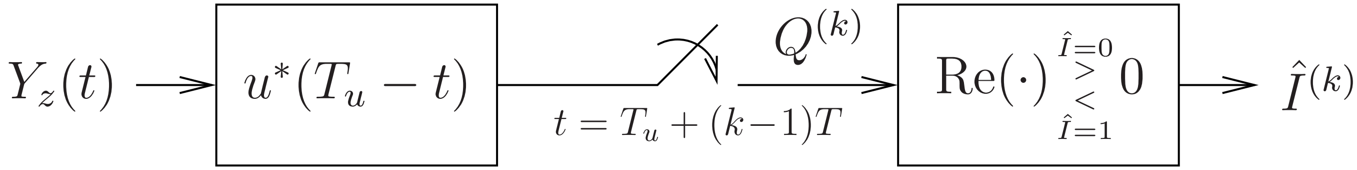
Could be called "orthogonal time-division multiplexing."

$$X_z(t) \;=\; \sum_{l=1}^{K_b} D_z^{(l)} \sqrt{E_b}\, u\big(t-(l-1)T\big)$$

using BPSK $D_z^{(l)}$ as before. (Note: $T_p = T_u + (K_b - 1)T$.) The pulse waveform $u(t)$ is orthogonal to its T-shifts.



Binary stream demodulator:



for $k \in \{1, \ldots, K_b\}$.

BEP identical to that of BPSK. Spectral efficiency is

$$\left.\begin{array}{l} W_b = \frac{1}{T} \text{ bits/sec,} \\ B_T \geq \frac{1}{T} \text{ Hz} \end{array}\right\} \Rightarrow \eta_B = \frac{W_b}{B_T} \leq 1 \text{ bit/sec/Hz}$$

Note: In practice, Linear/OFDM/OCDM modulations are combined with stream modulation.

Combined Orthogonal & Linear Modulation:

Say we have K_b bits to send over L orthogonal waveforms:

$$X_z(t) = \sum_{l=1}^{L} D_z^{(l)} \sqrt{E_b} s^{(l)}(t)$$

where $\{s^{(l)}(t)\}_{l=1}^{L}$ are orthonormal and $D_z^{(l)}$ is $2^{\frac{K_b}{L}}$-ary for each l. We will assume that

$$\mathrm{E}\{D_z^{(l)} D_z^{(k)*}\} = \begin{cases} \frac{K_b}{L} & k = l \\ 0 & k \neq l \end{cases}$$

which means that the symbols used on different waveforms are uncorrelated. It also guarantees an energy-per-bit of E_b.

Spectral Characteristics:

$$
\begin{aligned}
D_{X_z}(f) &= \frac{1}{K_b}\,\mathrm{E}\Big\{\Big|\int_{-\infty}^{\infty} X_z(t)e^{-j2\pi f t}dt\Big|^2\Big\} \\
&= \frac{1}{K_b}\,\mathrm{E}\Big\{\Big|\int_{-\infty}^{\infty}\sum_{l=1}^{L} D_z^{(l)}\sqrt{E_b}s^{(l)}(t)e^{-j2\pi f t}dt\Big|^2\Big\} \\
&= \frac{E_b}{K_b}\,\mathrm{E}\Big\{\Big|\sum_{l=1}^{L} D_z^{(l)}S^{(l)}(f)\Big|^2\Big\} \\
&= \frac{E_b}{K_b}\sum_{l=1}^{L}\sum_{k=1}^{L}\mathrm{E}\big\{D_z^{(l)}D_z^{(k)*}\big\}S^{(l)}(f)S^{(k)*}(f) \\
&= \frac{E_b}{L}\sum_{l=1}^{L} G_{S^{(l)}}(f)
\end{aligned}
$$

WEP Analysis:

Since we assume an identical constellation on each waveform,

$$\mathsf{WEP} = 1 - \left(1 - \mathsf{WEP}^{(l)}\right)^{L}$$

where $\mathsf{WEP}^{(l)}$ denotes the per-waveform WEP.

Example 1: $2^{\frac{K_b}{L}}$-ary Stream Modulation

Time-multiplexing of L symbols with K_b/L bits per symbol:

$$X_z(t) \;=\; \sum_{l=1}^{L} D_z^{(l)} \sqrt{E_b}\, u\big(t-(l-1)T\big).$$

As before, the pulse waveform $u(t)$ is orthogonal to its T-shifts. But now $T_p = T_u + (L-1)T$ and $D_z^{(l)}$ is a symbol from a generic $2^{\frac{K_b}{L}}$-ary constellation (e.g., QAM, PAM, PSK).

Example 2: $2^{\frac{K_b}{L}}$-ary OFDM

L subcarriers with K_b/L bits per subcarrier:

$$s^{(l)}(t) = \begin{cases} \sqrt{\frac{1}{T_p}} \exp\big(j2\pi f_d(2l - L - 1)t\big) & t \in [0, T_p] \\ 0 & t \notin [0, T_p] \end{cases}$$

$$X_z(t) = \sum_{l=1}^{L} \underbrace{D_z^{(l)} \sqrt{E_b} s^{(l)}(t)}_{x^{(l)}_{I^{(l)}}(t)}$$

Here, $D_z^{(l)}$ is a symbol from a generic $2^{\frac{K_b}{L}}$-ary constellation (e.g., QAM, PAM, PSK).

For example, there are several ways to transmit 6 bits using OFDM:

- 6 sub-carriers with BPSK and $f_d = \frac{1}{4T_p}$
- 3 sub-carriers with QPSK and $f_d = \frac{1}{2T_p}$
- 2 sub-carriers with 8-PSK and $f_d = \frac{1}{2T_p}$

What do you expect for the spectral efficiencies?
What about the relative WEP performance?

Example 3: Streamed M-ary OFDM

In modern practical systems, the concepts of time multiplexing (i.e., streaming), frequency multiplexing, and linear modulation are often combined.

For example, a 1 Mbit block could be transmitted using 1024 consecutive OFDM frames of 256 subcarriers with 16-QAM (i.e., 4 bits) on each subcarrier.