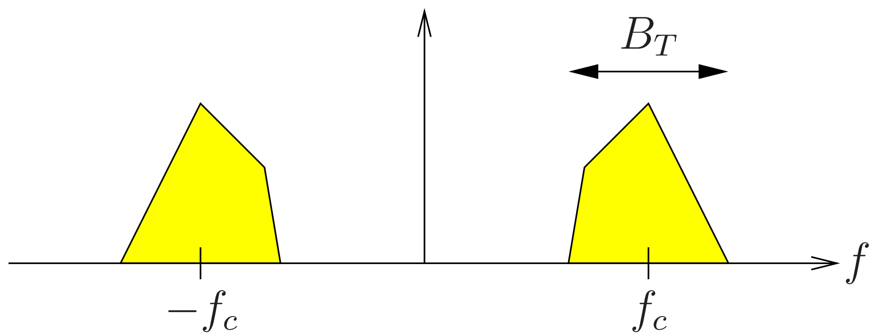
Review:

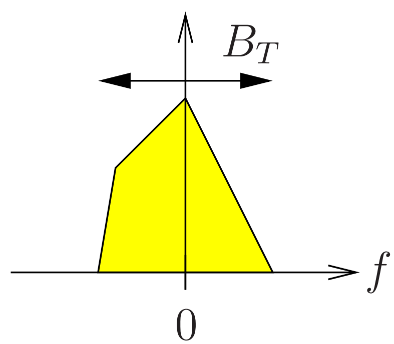
1. Complex baseband representations.
2. Random variables and random processes.
3. Additive noise model.

Complex Baseband Representations [Ch. 4]:

Many systems transmit real-valued passband signals:



but modem processing is done at baseband. Hence, a complex baseband signal representation is very useful.

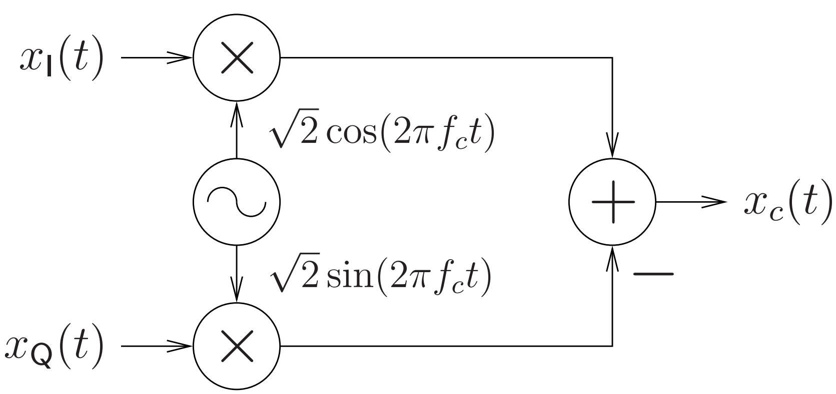


Baseband signal ("complex envelope"):

$$x_z(t) = x_{\mathsf{I}}(t) + j x_{\mathsf{Q}}(t) \quad \begin{cases} x_{\mathsf{I}}(t) \in \mathbb{R} & \text{"in phase"} \\ x_{\mathsf{Q}}(t) \in \mathbb{R} & \text{"quadrature"} \end{cases}$$

Conversion to passband signal $x_c(t)$:

$$\begin{aligned} x_c(t) &= \sqrt{2}\,\mathbb{R}\left[x_z(t) e^{j2\pi f_c t}\right] \\ &= \sqrt{2}\left[x_{\mathsf{I}}(t)\cos(2\pi f_c t) - x_{\mathsf{Q}}(t)\sin(2\pi f_c t)\right] \end{aligned}$$



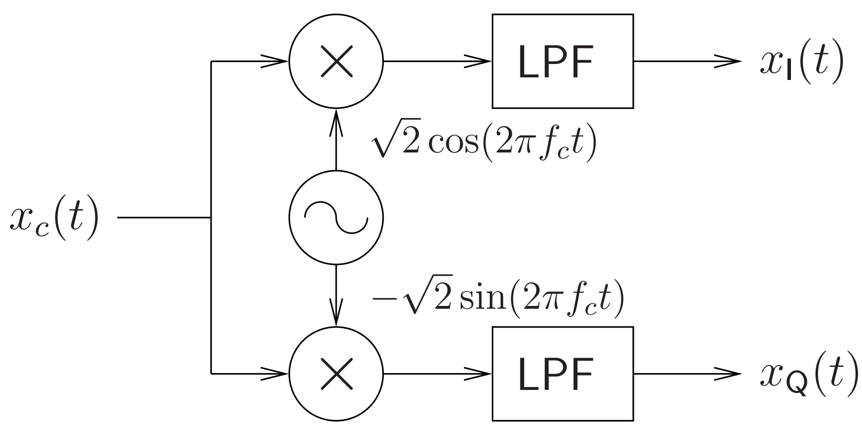
"quadrature modulator"

"I/Q upconverter"

Conversion from passband to baseband:

$$\begin{aligned} x_c(t)\sqrt{2}\cos(2\pi f_c t) &= x_{\mathsf{I}}(t) + x_{\mathsf{I}}(t)\cos(4\pi f_c t) \\ &\quad - x_{\mathsf{Q}}(t)\sin(4\pi f_c t) \\ -x_c(t)\sqrt{2}\sin(2\pi f_c t) &= x_{\mathsf{Q}}(t) - x_{\mathsf{Q}}(t)\cos(4\pi f_c t) \\ &\quad - x_{\mathsf{I}}(t)\sin(4\pi f_c t) \end{aligned}$$

LPF to remove double-frequency terms:



"quadrature demodulator"

"I/Q downconverter"

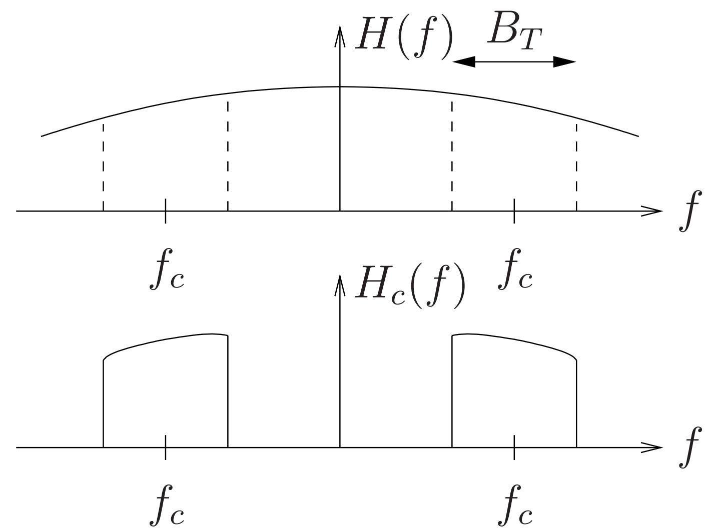
Signal spectra:

$$\begin{aligned} X_z(f) &:= \mathcal{F}\{x_z(t)\} && \text{Fourier transform} \\ G_{X_z}(f) &:= |X_z(f)|^2 && \text{“Energy spectrum”} \end{aligned}$$

Note that

$$G_{X_c}(f) = \frac{1}{2}G_{X_z}(f - f_c) + \frac{1}{2}G_{X_z}(-f - f_c)$$

Filtering of bandpass $X_c(f)$:

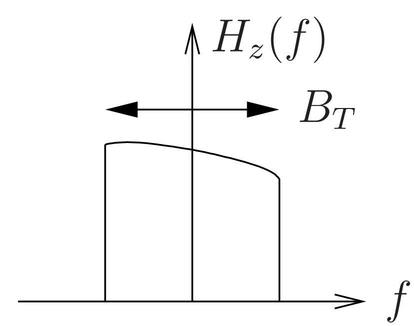


$$Y_c(f) = H(f)X_c(f),$$
$$\Rightarrow Y_c(f) = H_c(f)X_c(f)$$

via "bandpass equivalent" $H_c(f)$.

Translate filter to baseband:

$$h_c(t) = 2\mathbb{R}\left[h_z(t)e^{j2\pi f_c t}\right]$$
$$h_z(t) = h_{\mathsf{I}}(t) + jh_{\mathsf{Q}}(t)$$

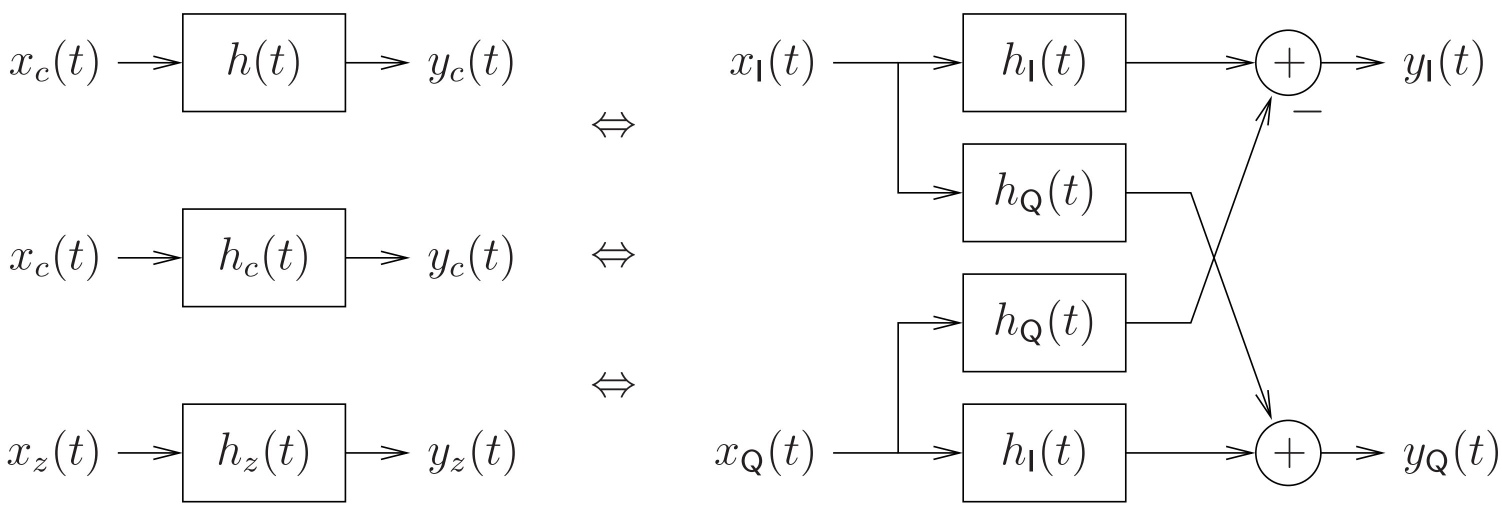


to get "baseband equivalent" $H_z(f)$.

Applying the baseband filter to a baseband signal is equivalent to applying the passband filter to a passband signal:

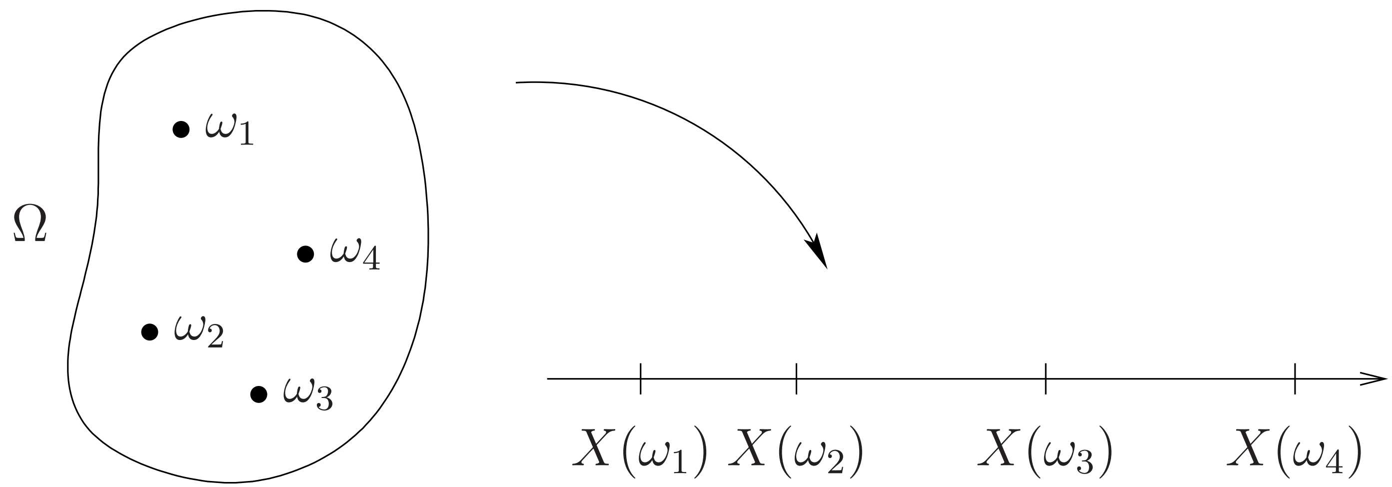
$$Y_c(f) = H_c(f)X_c(f) \quad \Leftrightarrow \quad Y_z(f) = H_z(f)X_z(f).$$

In other words, these are all equivalent:



Random Variables [Ch. 3]:

A RV $X(\omega)$ maps the sample space Ω to a real number:



Usually we use the shorthand notation X for the RV.

The value taken by a RV in a particular experiment is called a "sample" or "realization."

The cumulative distribution function (CDF) of RV $X(\omega)$ is

$$F_X(x) = \Pr\{\omega : X(\omega) \leq x\},$$

or, in shorthand notation,

$$F_X(x) = \Pr\{X \leq x\}.$$

Note:

$$\begin{aligned} F_X(-\infty) &= 0 \\ F_X(\infty) &= 1 \\ F_X(x) &= \text{increasing in } x. \end{aligned}$$

The probability density function (PDF) of $X(\omega)$

$$f_X(x) = \frac{d}{dx} F_X(x).$$

Discrete RVs don't have PDFs, but rather probability mass functions (PMFs)

$$p_X(x) = \Pr\{X = x\}.$$

We will use $p_X(x)$ for both PDFs and PMFs (unless there is a possibility of confusion).

Some properties of the PDF:

$$
\begin{aligned}
f_X(x) &\geq 0 \\
\int_{-\infty}^{\infty} f_X(\beta) d\beta &= 1 \\
\int_{-\infty}^{x} f_X(\beta) d\beta &= F_X(x) \\
\int_{x_1}^{x_2} f_X(\beta) d\beta &= \Pr\{x_1 < X \leq x_2\}
\end{aligned}
$$

Statistics of a RV:

Mean:

$$E(X) \;=\; \int_{-\infty}^{\infty} x\, p_X(x) dx \;=\; m_X$$

Variance:

$$E((X - m_X)^2) \;=\; \int_{-\infty}^{\infty} (x - m_X)^2\, p_X(x) dx \;=\; \sigma_X^2$$

In general:

$$E(g(X)) \;=\; \int_{-\infty}^{\infty} g(x)\, p_X(x) dx$$

Gaussian (or "normal") RV:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left[-\frac{(x-m_X)^2}{2\sigma_X^2}\right]$$

$$X \sim \mathcal{N}(m_X, \sigma_X^2)$$

Though the Gaussian CDF has no closed-form expression, the erf function is frequently tabulated.

$$\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt = 1 - \text{erfc}(z).$$

$$F_X(x) = \frac{1}{2} + \frac{1}{2}\text{erf}\left(\frac{x-m_X}{\sqrt{2}\sigma_X}\right)$$

Joint CDF:

$$F_{XY}(x,y) \;=\; \Pr\{X \le x, Y \le y\}$$

Joint PDF:

$$f_{XY}(x,y) \;=\; \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y)$$

Joint PMF:

$$p_{XY}(x,y) \;=\; \Pr\{X = x, Y = y\}$$

Conditional PDF of Y given that $X = x$:

$$p_{Y|X}(y \,|\, X = x) \;=\; \frac{p_{XY}(x,y)}{p_X(x)}$$

Total probability:

$$p_Y(y) = \int_{-\infty}^{\infty} p_{XY}(x,y)dx$$

Bayes rule:

$$p_{Y|X}(y \,|\, X=x) = \frac{p_{X|Y}(x \,|\, Y=y)p_Y(y)}{p_X(x)}$$

RVs X and Y are independent (i.e., $X \perp\!\!\!\perp Y$) when

$$\begin{aligned} p_{XY}(x,y) &= p_X(x)p_Y(y) \\ \Leftrightarrow\ p_{Y|X}(y \,|\, X=x) &= p_Y(y) \end{aligned}$$

Joint Statistics of Two RVs:

cross-correlation:

$$\mathrm{E}[XY] = \int\int x\, y\, p_{XY}(x,y) dx dy$$

cross-covariance:

$$\mathrm{E}[(X - m_X)(Y - m_Y)] = \int\int (x - m_X)(y - m_Y)\, p_{XY}(x,y) dx dy$$

In general:

$$E[g(X,Y)] = \int\int g(x,y) p_{XY}(x,y) dx dy$$

Gaussian random vector (i.e., jointly Gaussian RVs):

$$\underline{N} = [N_1, \ldots, N_L]^T$$

Joint pdf:

$$f_N(\underline{n}) = \frac{1}{\sqrt{(2\pi)^L \det \boldsymbol{C}_N}} \exp\left[-\frac{1}{2}(\underline{n} - \underline{m}_N)^T \boldsymbol{C}_N^{-1} (\underline{n} - \underline{m}_N)\right]$$

with mean vector

$$\underline{m}_N = \mathrm{E}(\underline{N})$$

and covariance matrix

$$\boldsymbol{C}_N = \mathrm{E}[(\underline{N} - \underline{m}_N)(\underline{N} - \underline{m}_N)^T]$$

Complex Gaussian RV:

$$Z = N_{\mathsf{I}} + jN_{\mathsf{Q}} \quad \Leftrightarrow \quad \begin{pmatrix} N_{\mathsf{I}} \\ N_{\mathsf{Q}} \end{pmatrix} = \underline{N}$$

where

$$\text{Circular} \quad \Leftrightarrow \quad \boldsymbol{C}_N = \begin{pmatrix} \frac{1}{2}\sigma_Z^2 & 0 \\ 0 & \frac{1}{2}\sigma_Z^2 \end{pmatrix}$$

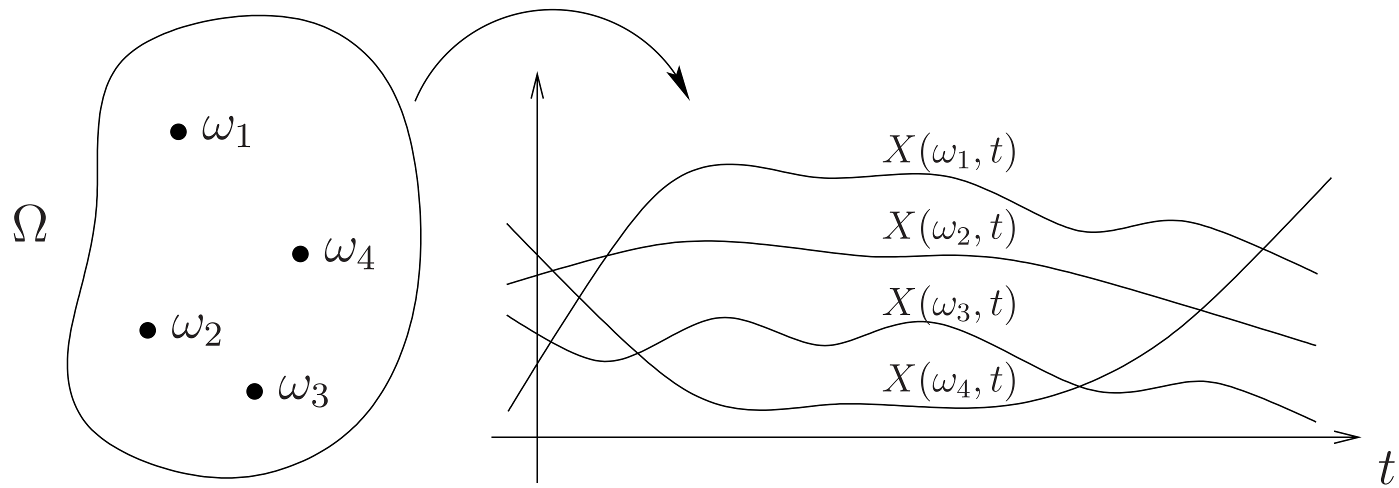
Can write PDF as

$$f_Z(z) = \frac{1}{\pi\sigma_Z^2} \exp\left[-\frac{|z - m_Z|^2}{\sigma_Z^2}\right]$$

$$Z \sim \mathcal{CN}(m_Z, \sigma_Z^2)$$

Random Processes [Ch. 9]:

A RP $X(\omega, t)$ maps the sample space Ω to a signal:



Usually we use the shorthand notation $X(t)$ for the RP.

The waveform taken by a RP in a particular experiment is called a "sample path" or "realization."

Properties:

A sample of a RP (e.g., $X(0)$) is a RV.

A RP is stationary if the joint PDF of any set of samples is invariant to bulk sampling-time shifts:

$$\begin{aligned} & f_{N(t_0),N(t_1),\dots,N(t_M)}(n_1,n_2,\dots,n_M) \\ &= f_{N(t_0+\tau),N(t_1+\tau),\dots,N(t_M+\tau)}(n_1,n_2,\dots,n_M), \quad \forall\, t_1,\dots,t_M,\tau \end{aligned}$$

A RP is wide-sense stationary (WSS) if at least the mean and autocorrelation are invariant to time shifts:

$$\begin{aligned} E[N(t_1)] &= E[N(t_2)], \quad \forall\, t_1,t_2 \\ E[N(t_1)N(t_1-\tau)] &= E[N(t_2)N(t_2-\tau)], \quad \forall\, t_1,t_2,\tau \end{aligned}$$

For a WSS RP we have the (time-invariant) statistics

$$\begin{aligned}\text{mean:} \quad m_N &= E[N(t)]\\ \text{autocorrelation:} \quad R_N(\tau) &= E[N(t)N(t-\tau)]\end{aligned}$$

Note that $\sigma_N^2 = R_N(0) - m_N^2$ and $R_N(\tau) = R_N(-\tau)$.

A Gaussian RP is one where any collection of samples is composed of jointly Gaussian RVs.

A stationary Gaussian RP is completely described by its mean m_N and autocorrelation $R_N(\tau)$.

From here on, we assume zero-mean processes!

Power spectral density (PSD) of a WSS RP:

$$S_N(f) = \lim_{T\to\infty} \frac{1}{2T} \mathrm{E}\left[\left|\int_{-T}^{T} N(t)e^{-j2\pi ft}dt\right|^2\right]$$

or

$$S_N(f) = \int_{-\infty}^{\infty} R_N(\tau)e^{-j2\pi f\tau}d\tau$$

Note that

$$S_N(f) \in \mathbb{R}, \quad S_N(f) \geq 0, \quad \text{and } S_N(f) = S_N(-f),$$

and also that

$$\sigma_N^2 = R_N(0) = \int_{-\infty}^{\infty} S_N(f)e^{j2\pi f0}df = \int_{-\infty}^{\infty} S_N(f)df.$$

A white RP has a constant PSD. For example, we will model white noise $W(t)$ via the "two-sided PSD" (p. 9.20)

$$S_W(f) = \frac{N_0}{2},$$

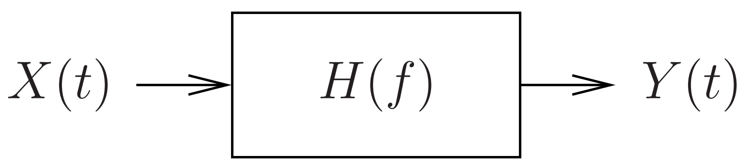
implying

$$\begin{aligned} R_W(\tau) &= \frac{N_0}{2}\delta(\tau) \\ \sigma_W^2 &= R_W(0) = \int_{-\infty}^{\infty} S_N(f)df = \infty \end{aligned}$$

Note: thermal noise is approximately constant for $|f| < 10^{12}$ Hz, so we often approximate it as white noise.

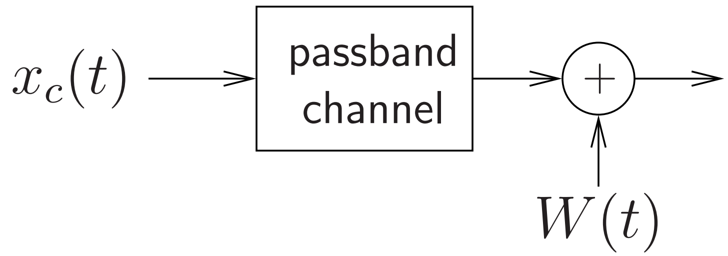
Linear filtering of RPs:

- A linear combination of Gaussian RVs is a Gaussian RV.
- Linear filtering of a Gaussian RP yields a Gaussian RP.
- LTI filtering of a stationary RP yields a stationary RP.



$$S_Y(f) = |H(f)|^2 S_X(f)$$

The Additive Noise Model [Ch. 10]:



We assume $W(t)$ is zero-mean stationary Gaussian with

$$\begin{aligned} R_W(\tau) &= \mathrm{E}\{W(t)W(t-\tau)\} \quad \text{autocorrelation} \\ S_W(f) &= \mathcal{F}\{R_W(\tau)\} \quad \text{power spectrum} \end{aligned}$$

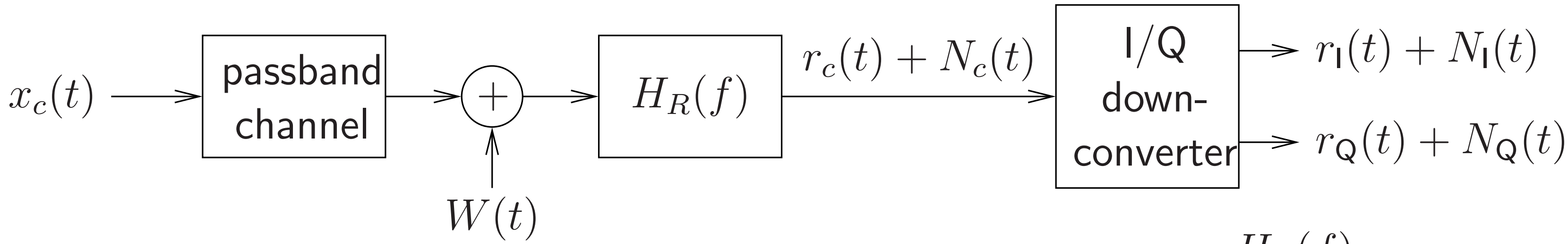
We also assume a constant PSD (i.e., white noise):

$$S_W(f) = N_0/2$$

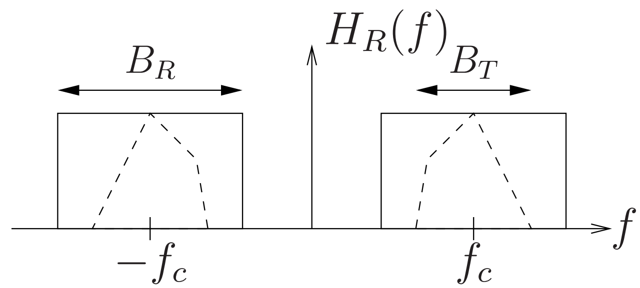
Thus

$$R_W(\tau) = \frac{N_o}{2}\delta(\tau), \qquad \sigma_W^2 = R_W(0) = \infty$$

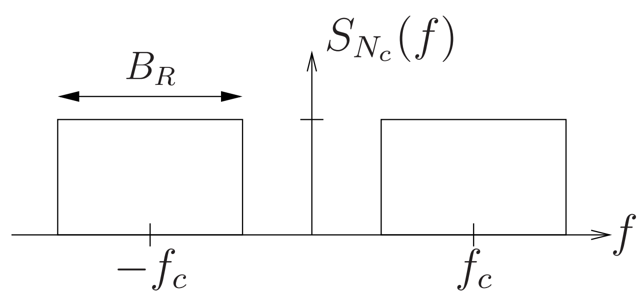
Received noise model:



Here $H_R(f)$ is the receive filter:



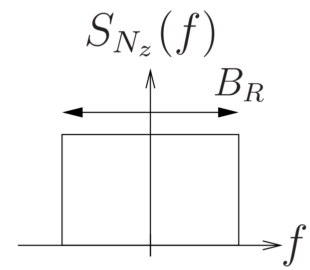
The passband noise spectrum is



$$S_{N_c}(f) = \frac{N_o}{2}|H_R(f)|^2$$

Baseband equivalent noise model:

- Say $\begin{cases} N_c(t) &= \sqrt{2}\mathbb{R}\left[N_z(t)e^{j2\pi f_c t}\right] \\ N_z(t) &= N_{\mathsf{I}}(t) + jN_{\mathsf{Q}}(t) \end{cases}$



$$S_{N_c}(f) = \tfrac{1}{2}S_{N_z}(f-f_c) + \tfrac{1}{2}S_{N_z}(-f-f_c)$$

- Fitz shows that $N_{\mathsf{I}}(t)$ and $N_{\mathsf{Q}}(t)$ are zero-mean, jointly stationary and jointly Gaussian with

$$R_{N_{\mathsf{I}}}(\tau) = R_{N_{\mathsf{Q}}}(\tau) \quad \text{and} \quad R_{N_{\mathsf{I}}N_{\mathsf{Q}}}(\tau) = -R_{N_{\mathsf{I}}N_{\mathsf{Q}}}(-\tau)$$

- Thus $R_{N_{\mathsf{I}}N_{\mathsf{Q}}}(0) = 0 \quad \Rightarrow \quad N_{\mathsf{I}}(t_o) \perp\!\!\!\perp N_{\mathsf{Q}}(t_o)$ for any t_o and $S_{N_z}(f) = 2\underbrace{S_{N_{\mathsf{I}}}(f)}_{\text{even}} - j2\underbrace{S_{N_{\mathsf{I}}N_{\mathsf{Q}}}(f)}_{\text{odd}}$.

Complex white noise model:

- With flat, unity-gain receive filter and $B_R > B_T$, we often approximate $N_z(t)$ by circular complex Gaussian noise $W_z(t)$ with statistics given by

$$S_{W_z}(f) = N_o \quad \Leftrightarrow \quad R_{W_z}(\tau) = N_o \delta(\tau)$$