Communication of Multiple Bits: [Ch. 14]

Problem Setup:

- K_b bits $\quad\Rightarrow\quad M = 2^{K_b}$ waveforms

  $\underline{I} \in \{0, 1, 2, \ldots, M-1\}$ "information word"

  $P(\underline{I} = i) \stackrel{\Delta}{=} \pi_i$ "prior probability"

- $\underline{I} = i \quad\Rightarrow\quad X_z(t) = x_i(t)$ "modulation"

  $x_i(t)$ has time support only on $[0, T_p]$

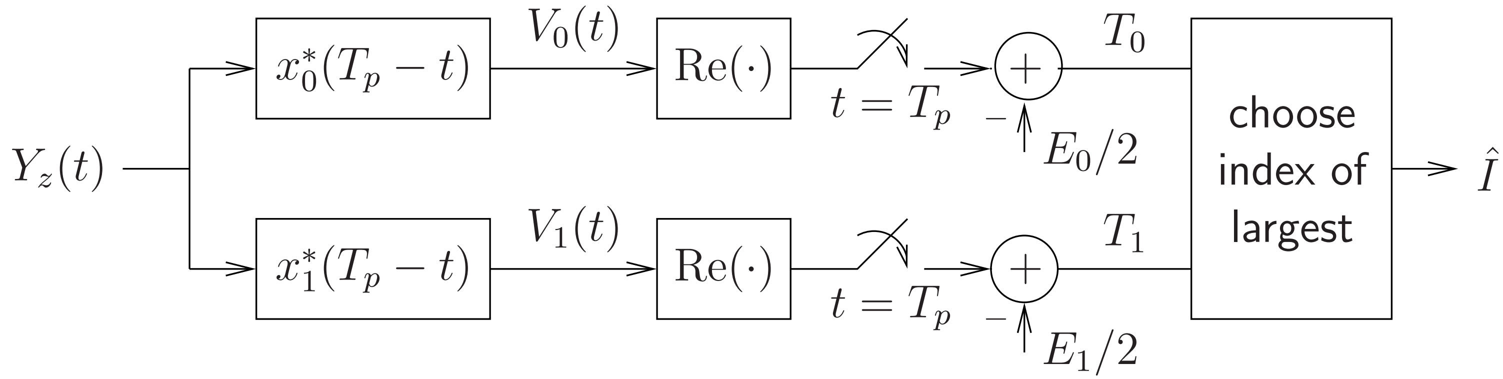
  $W_b = K_b/T_p$ bits/sec "bit rate"

  $E_b = \frac{1}{K_b}\sum_{i=0}^{M-1} \pi_i E_i$ "average energy per bit"

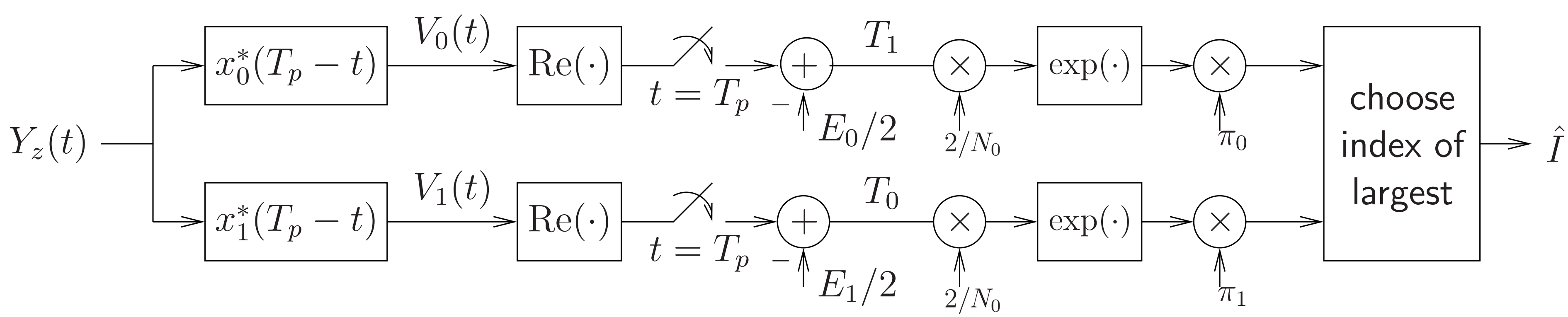
- $Y_z(t) = X_z(t) + W_z(t)$ "AWGN channel"

  $W_z(t)$ is circular complex Gaussian with $S_{W_z}(f) = N_0$

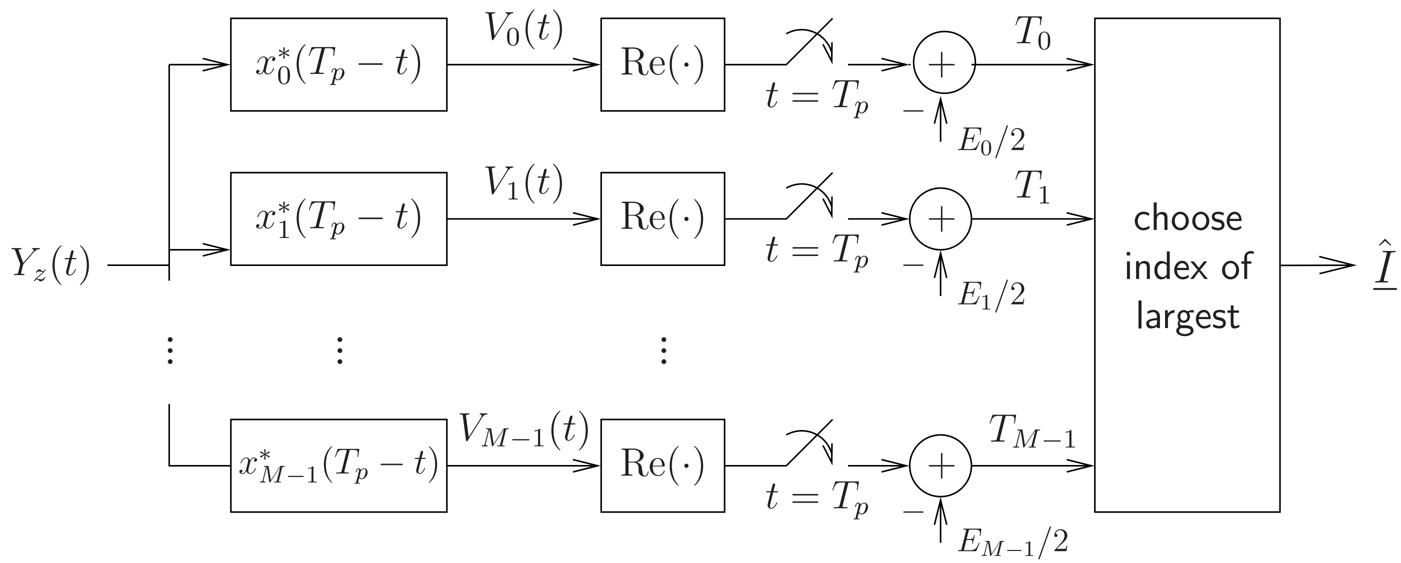
Recall binary ML demodulator (optimum for equal priors):



and binary MAP demodulator (optimum for general case):



Can generalize to the M-ary equal-priors case:

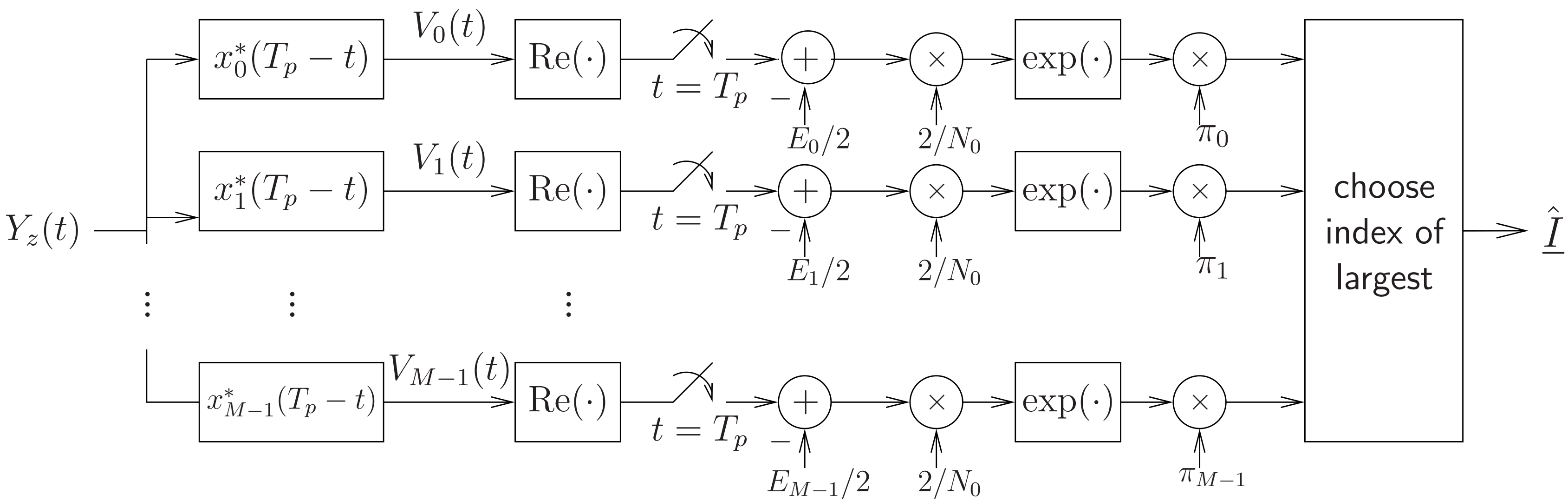


$$V_i(t) \triangleq Y_z(t) * x_i^*(T_p - t) = \int_{-\infty}^{\infty} Y_z(\tau) x_i^*(T_p - t + \tau) d\tau.$$

$$T_i \triangleq \operatorname{Re} \int_{-\infty}^{\infty} Y_z(\tau) x_i^*(\tau) d\tau - \frac{E_i}{2},$$ "*ML metric*"

$$\underline{\hat{I}} = \arg \max_{i \in \{0,\ldots,M-1\}} T_i,$$ "*ML word demodulator* (MLWD)"

Can also generalize to the M-ary unequal-priors case:



$$\hat{\underline{I}} = \arg \max_{i \in \{0,\ldots,M-1\}} \exp\left(\frac{2T_i}{N_0}\right) \pi_i.$$

"*MAP word demodulator* (MAPWD)"

Word Error Probability (WEP) Analysis:

$$\text{WEP} \triangleq \sum_{j=0}^{M-1} \Pr(\hat{\underline{I}} \neq j | \underline{I} = j)\pi_j$$

Non-linearity of MAPWD makes WEP difficult to analyze.

So consider equal priors (i.e., $\pi_i = \frac{1}{M}\ \forall i$) and MLWD.

$$\text{WEP} = \frac{1}{M} \sum_{j=0}^{M-1} \Pr(\hat{\underline{I}} \neq j | \underline{I} = j)$$

$$\Pr(\hat{\underline{I}} \neq j | \underline{I} = j) = \Pr\big(\exists i \neq j \text{ s.t. } T_i > T_j | \underline{I} = j\big).$$

What are the statistics of $T_i\big|_{\underline{I}=j}$?

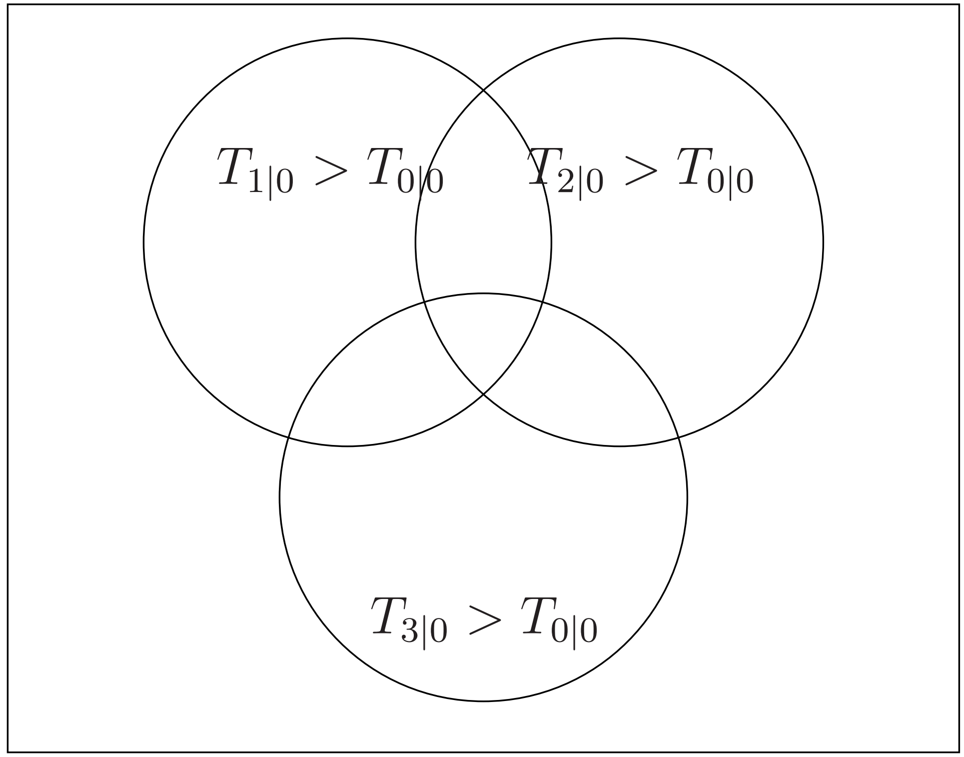
Consider the *conditional likelihood statistic* $T_{i|j} \triangleq T_i\big|_{\underline{I}=j}$.

$$T_{i|j} = \mathrm{Re} \int_{-\infty}^{\infty} \big[x_j(t) + W_z(t)\big] x_i^*(t) dt - \frac{E_i}{2}$$
$$= \sqrt{E_i E_j}\, \mathrm{Re}\, \rho_{ji} + N_{\mathsf{I}}^{(i)} - \frac{E_i}{2}$$

where $N_{\mathsf{I}}^{(i)} \sim \mathcal{N}(0, E_i N_0/2)$. Then using the *union bound*,

$$\begin{aligned} \Pr(\hat{\underline{I}} \neq j | \underline{I} = j) &= \Pr(\exists i \neq j \text{ s.t. } T_{i|j} > T_{j|j}) \\ &= \Pr(\cup_{i=0, i\neq j}^{M-1} \{T_{i|j} > T_{j|j}\}) \\ &\leq \sum_{i=0, i\neq j}^{M-1} \underbrace{\Pr(T_{i|j} > T_{j|j})}_{\text{pairwise error prob.}} \end{aligned}$$

Illustration of union bound (for $j = 0$, $M = 4$):



Note: At high SNR, unusual for more than one ML metric $T_{i|j}$ to exceed $T_{j|j}$. So, union bound is tight at high SNR.

Now let's analyze the *pairwise error probability* (PWEP):

$$\begin{aligned}
\Pr(T_{i|j} > T_{j|j}) &= \Pr\left(\sqrt{E_i E_j}\,\mathrm{Re}\,\rho_{ji} - \frac{E_i}{2} + N_{\mathsf{I}}^{(i)} > \frac{E_j}{2} + N_{\mathsf{I}}^{(j)}\right) \\
&= \Pr\Bigg(\underbrace{N_{\mathsf{I}}^{(i)} - N_{\mathsf{I}}^{(j)}}_{N_{ij}} > \underbrace{\frac{E_i + E_j}{2} - \sqrt{E_i E_j}\,\mathrm{Re}\,\rho_{ji}}_{\Delta_E(i,j)/2}\Bigg) \\
N_{ij} &= \mathrm{Re} \int_{-\infty}^{\infty} W_z(t)\left[x_i^*(t) - x_j^*(t)\right] dt \\
\sigma_{N_{ij}}^2 &= \frac{N_0}{2}\Delta_E(i,j) \\
\Pr(T_{i|j} > T_{j|j}) &= \frac{1}{2} - \frac{1}{2}\,\mathrm{erf}\left(\frac{\Delta_E(i,j)/2}{\sqrt{2}\sigma_{N_{ij}}}\right) = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{\Delta_E(i,j)}{4N_0}}\right)
\end{aligned}$$

which is a direct extension of binary BEP.

Plugging back into union bound, we find

$$\text{WEP} \le \frac{1}{M}\sum_{j=0}^{M-1}\sum_{i=0,i\neq j}^{M-1}\frac{1}{2}\operatorname{erfc}\left(\sqrt{\frac{\Delta_E(i,j)}{4N_0}}\right).$$

- Since, at high SNR, $\operatorname{erfc}(x) \approx \exp(-x^2)$, WEP dominated by the smallest $\Delta_E(i,j)$.
- Frequency of different $\Delta_E(i,j)$: "*distance spectrum*."
  $A_d(k) \triangleq$ number of signal pairs having distance $\Delta_E(k)$.

$$\text{WEP} \approx \frac{A_d(\min)}{2M}\operatorname{erfc}\left(\sqrt{\tfrac{\Delta_E(\min)}{4N_0}}\right) \quad \text{at high SNR.}$$

- Design strategy: Maximize $\Delta_E(\min)$.
- Designs with equal $\Delta_E(i,j)$: "*geometrically uniform*."

Example 1: M-ary FSK:

$$x_i(t) = \begin{cases} \sqrt{\frac{K_b E_b}{T_p}} \exp\big(j2\pi f_d (2i - M + 1)t\big) & t \in [0, T_p] \\ 0 & t \notin [0, T_p] \end{cases}$$

where *frequency deviation* f_d is a design parameter.

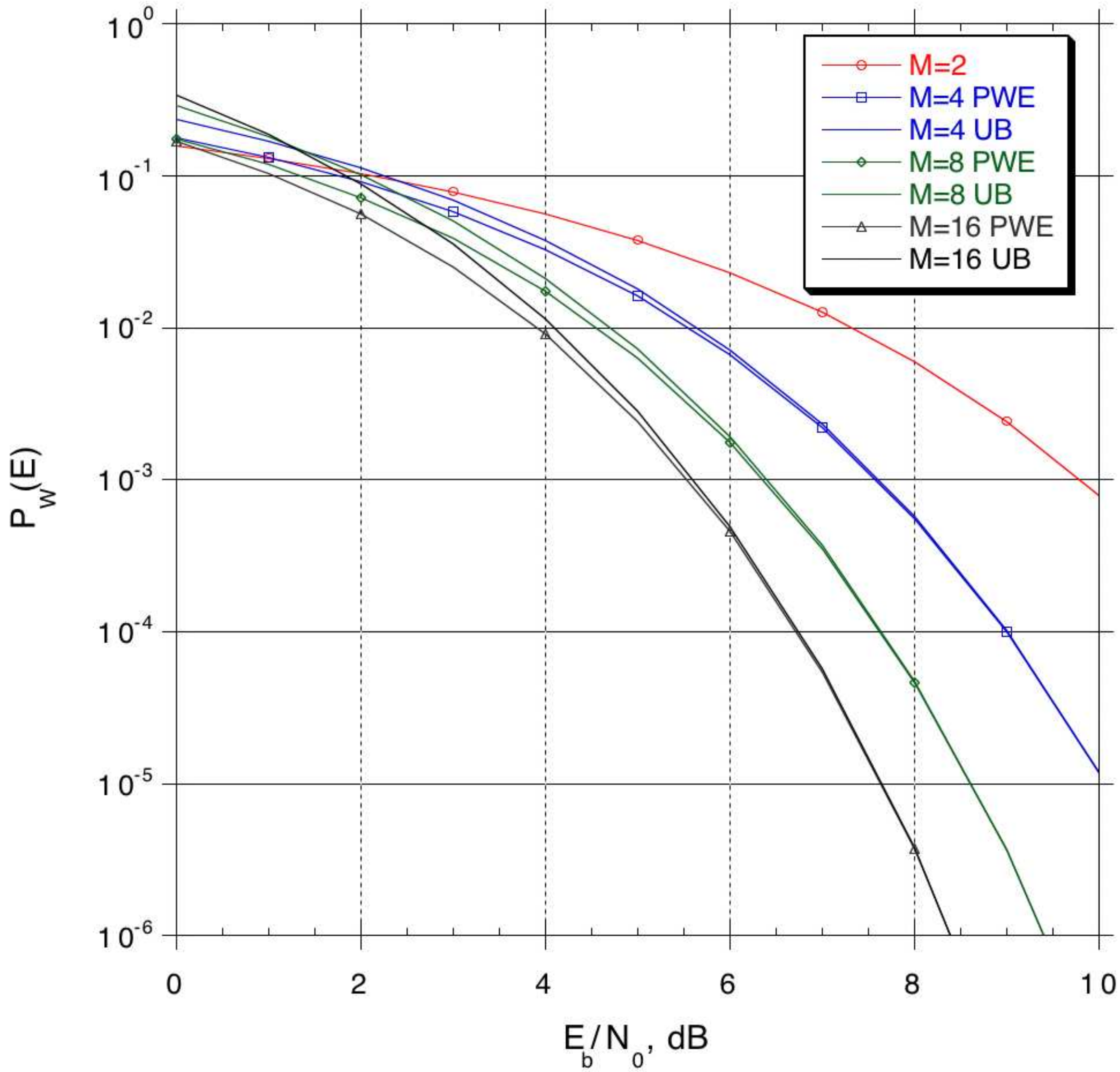
Orthogonal M-FSK:

$f_d = \frac{1}{4T_p} \rightsquigarrow \operatorname{Re} \rho_{ij}\big|_{i \neq j} = 0$; geometrically uniform.

Union Bound on WEP:

$$\Delta_E(i,j) = E_i + E_j = 2K_b E_b \;\Rightarrow\; \text{WEP} \leq \frac{M-1}{2} \operatorname{erfc}\left(\sqrt{\frac{K_b E_b}{2N_0}}\right)$$

Orthogonal M-FSK: Exact WEP and Union Bound



Spectral Efficiency of M-FSK:

For M-FSK,

$$G_{X_i}(f) = |X_i(f)|^2 = K_b E_b T_p \left(\frac{\sin(\pi[f - f_d(2i - M + 1)]T_p)}{\pi[f - f_d(2i - M + 1)]T_p} \right)^2$$

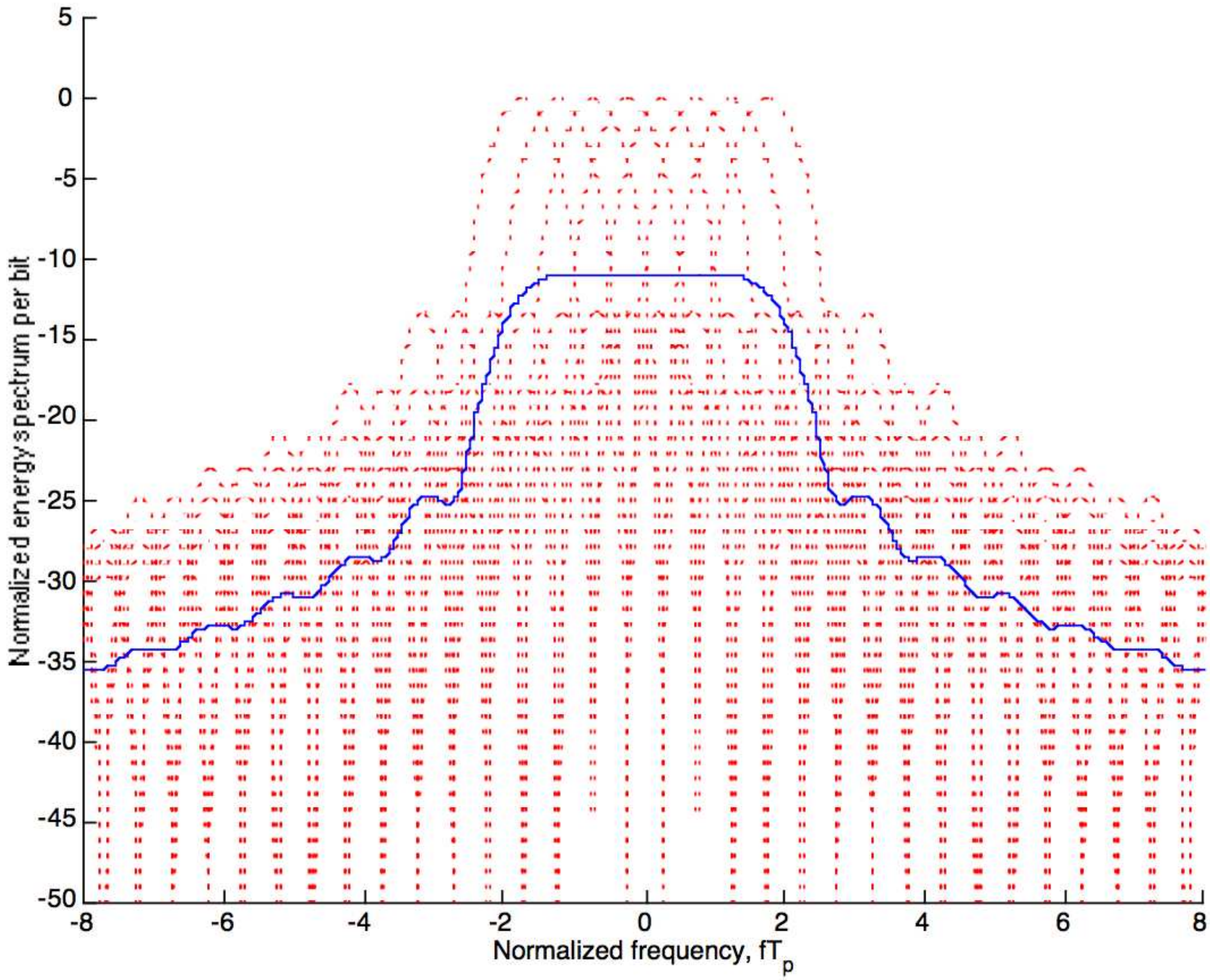
and

$$D_{X_z}(f) = \frac{1}{K_b} \sum_{i=0}^{M-1} \pi_i G_{X_i}(f).$$

Notice that $B_T \approx 2 f_d 2^{K_b}$, so that, with $f_d = 0.25/T_p$,

$$\eta_B = \frac{W_b}{B_T} \approx \frac{K_b/T_p}{2^{K_b-1}/T_p} = \frac{K_b}{2^{K_b-1}}.$$

8-FSK Example: $\{G_{X_i}(f)\}_{i=0}^{7}$ (dotted) and $D_{X_z}(f)$ (solid):



Orthogonal M-FSK Summary...

As K_b increases:

- reliability increases
- bandwidth efficiency decreases (exponentially)
- complexity increases (exponentially)

Hence, M-FSK useful when high reliability is required and when low spectral efficiency can be tolerated.

Complexity limits us to somewhat small values for K_b. (In practice, combine FSK with *stream modulation*.)

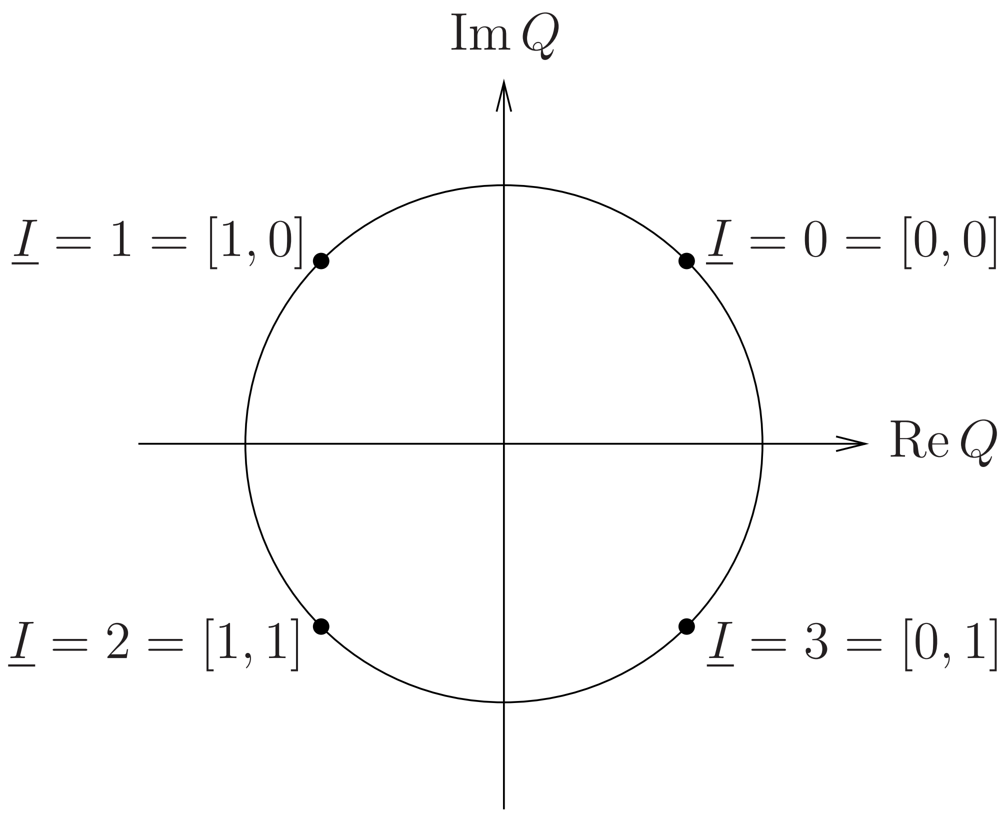
Example 2: M-ary PSK:

$$x_i(t) = \begin{cases} \sqrt{\frac{K_b E_b}{T_p}} \exp\left(j \frac{\pi(2i+1)}{M}\right) & t \in [0, T_p] \\ 0 & t \notin [0, T_p] \end{cases}$$

Can write as $x_i(t) = d_i u(t)$ with common pulse $u(t)$.

$$\begin{aligned} \underline{\hat{I}} &= \arg\max_i T_i \\ &= \arg\max_i \mathrm{Re}\left[e^{-j\theta_i} \underbrace{\int_{-\infty}^{\infty} Y_z(t) u^*(t) dt}_{\text{MF output } Q} \right] \text{ for } \theta_i = \frac{\pi(2i+1)}{M} \\ &= \arg\min_i \left| Q - e^{j\theta_i} \right|^2 \;=\; \arg\min_i |\angle Q - \theta_i| \end{aligned}$$

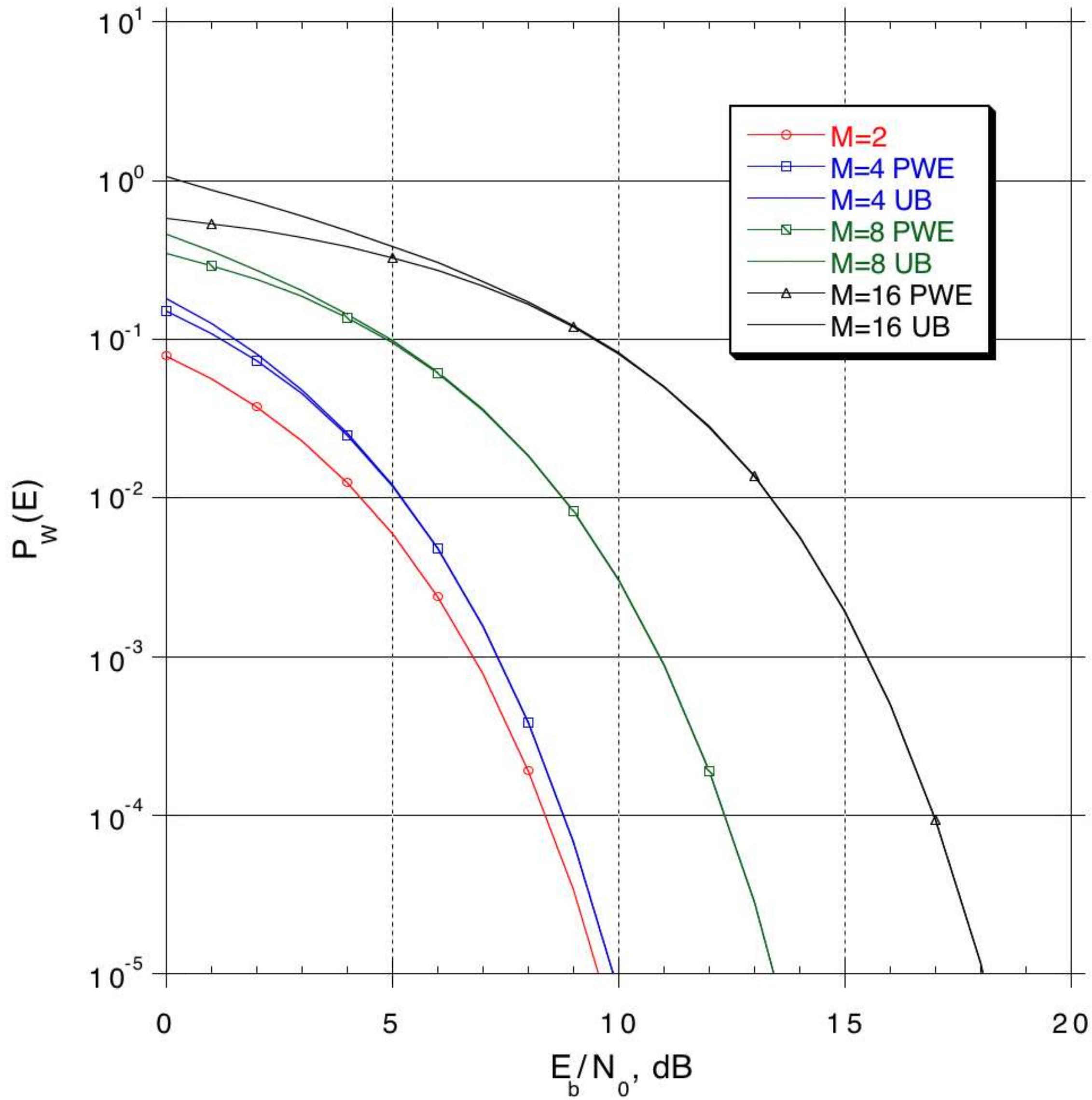
Decision regions for matched-filter output $Q \in \mathbb{C}$:



Gray mapping ensures one bit change for phase neighbors.

Note decoding of $I^{(0)}$ independent of $I^{(1)}$: parallel decoders!

M-PSK: Exact WEP and Union Bound



Spectral Efficiency of M-PSK:

For M-PSK,

$$G_{X_i}(f) \;=\; |X_i(f)|^2 \;=\; K_b E_b T_p \left(\frac{\sin(\pi f T_p)}{\pi f T_p}\right)^2,$$

so that

$$D_{X_z}(f) \;=\; \frac{1}{K_b} \sum_{i=0}^{M-1} \pi_i G_{X_i}(f) \;=\; E_b T_p \left(\frac{\sin(\pi f T_p)}{\pi f T_p}\right)^2.$$

Previously we found that $B_T \approx 1/T_p$, so that

$$\eta_B \;=\; \frac{W_b}{B_T} \;\approx\; \frac{K_b/T_p}{1/T_p} \;=\; K_b.$$

M-PSK Summary...

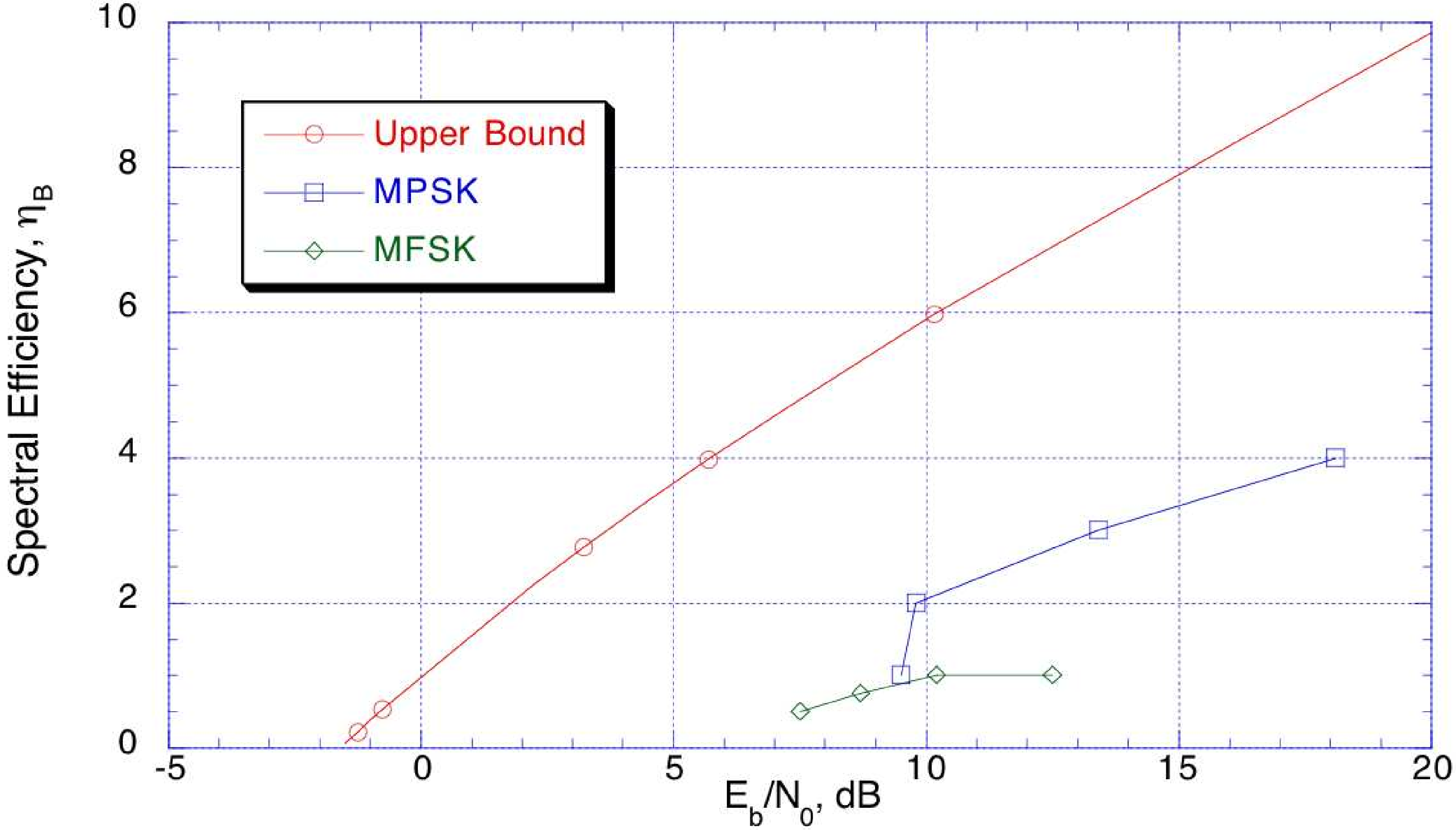
As K_b increases:

- reliability decreases
- bandwidth efficiency increases ($\eta_B = K_b$)
- complexity doesn't change much

Hence, M-PSK useful when high bandwidth efficiency is required and SNR is reasonably high.

Performance limits us to somewhat small values for K_b. (In practice, combine PSK with *stream modulation*.)

Comparison of spectral efficiencies to Shannon bound:



(Considering WEP$= 10^{-5}$ as "reliable.")